

PARAHORIC SPECIAL ORTHOGONAL, SYMPLECTIC AND SPIN BUNDLES ON A COMPACT RIEMANN SURFACE -I

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ABSTRACT. Let $p : Y \rightarrow X$ be a Galois cover of smooth projective curves over \mathbb{C} with Galois group π . This paper is devoted to the study of principal orthogonal and symplectic bundles E on Y to which the action of π on Y lifts. We notably describe them intrinsically in terms of objects defined on X and call these objects parahoric bundles. We give necessary and sufficient conditions for the non-emptiness of the moduli of stable and semi-stable parahoric special orthogonal, symplectic and spin bundles on the projective line \mathbb{P}^1 .

CONTENTS

1. Introduction	2
1.1. Remarks on Notation	5
1.2. Acknowledgements	5
2. Degenerate Orthogonal bundles with flags	6
3. Degenerate Special Orthogonal bundles with flags	15
3.1. Interpretation of π - SO_n bundles as parahoric bundles	16
4. Parahoric Symplectic bundles	18
5. Criterion of non-emptiness of parahoric moduli on \mathbb{P}^1 for $G = \mathrm{SO}_n, \mathrm{Sp}_{2n}$	20
5.1. (Semi)-stability conditions	20
5.2. Passage from Parahoric to Parabolic	21
5.3. Passage to generic bundles	25
5.4. Recall of Schubert states and Gromov–Witten numbers	26
5.5. Formulation of inequalities	28
5.6. Cross-checks	30
References	33

1. INTRODUCTION

Let X be a smooth projective curve over the field of complex numbers \mathbb{C} . Let $p : Y \rightarrow X$ be a Galois cover of smooth projective curves with Galois group π . By a π - G bundle, we mean a principal G -bundle E on Y such that the action of π lifts to E .

In the case $G = \mathrm{GL}_n$, following [11, Mehta-Seshadri] one knows that π - GL_n -bundles are described intrinsically on X as parabolic vector bundles, i.e., vector bundles on X together with a flag structure equipped with weights on some points of X .

In this paper we study this problem for the case of classical groups with the view towards obtaining more explicit intrinsic descriptions and using them to give necessary and sufficient conditions for non-emptiness of moduli on \mathbb{P}^1 . Our objective is to show how the results in [1, BS] can be seen more explicitly in terms of a vector bundle W equipped with a non-degenerate π -invariant quadratic form q' on some Galois cover. This approach is closer in spirit to Seshadri [18, CSS] and to Ramanan [14, R]. In particular since the group O_n is not simply connected and disconnected, the cases of these groups are not directly covered by [1] and [8].

We show in Theorem 2.0.14 that π - O_n -bundles can be described as *parabolic vector bundles* with weights symmetric about $1/2$ together with a quadratic structure: the underlying vector bundle V is endowed with a generically non-degenerate quadratic form q having “singularities of order at most one,” the underlying vector space at branch points is equipped with local quadratic structures and isotropic flags compatible with q (cf Definition 2.0.3). We call these bundles degenerate orthogonal bundles with flags.

A similar intrinsic description for π - SO_n bundles as degenerate orthogonal bundles with additional structure is also given. We then show that π - SO_n -bundles can also be interpreted as parahoric special orthogonal bundles in the sense of [1].

The concept of weights provides a candidate isomorphism identifying the local automorphism group $G(\mathcal{O}_y)^{\pi_y}$ of π - O_n -bundles with the local automorphism group of (quasi)-parahoric orthogonal bundles or what is the same with Bruhat–Tits group schemes.

Moreover the definition of stability and (semi)-stability of parahoric orthogonal bundles are described in terms of weights.

In the short section on symplectic bundles we state the important definitions and results in the symplectic case. The proofs are very similar to the orthogonal case, so we omit them.

In Theorems 5.5.1 and 5.5.2 and Remark 5.5.5, we show some inequalities in terms of Gromov–Witten numbers whose satisfaction is the necessary and sufficient condition for the existence of stable and polystable parahoric special orthogonal, symplectic and Spin bundles on \mathbb{P}^1 . These Gromov–Witten numbers are computable for any G/P where P is any parabolic subgroup (cf. [20] and Remark 5.5.3). To determine the existence, it is clear that we only need to check for only finitely many choices of degrees of sub-bundles as the weights are all positive and bounded. Thus, after finitely many computations, we need to check only finitely many inequalities to determine whether there exists a stable or semi-stable parahoric special orthogonal or symplectic bundle.

The strategy of proof is inspired by [3, P. Belkale]. The main new ingredients are Definitions 2.0.3, 4.0.4 and the passage from parahoric to parabolic bundles preserving (semi)-stability.

The Theorem 5.5.1 also solves the ‘multiplicative Horn problem’ or following the survey of Kostov [10] the ‘Deligne–Simpson Problem’: Let $\overline{A_i}$ be given conjugacy classes in $K_G = \mathrm{SO}_n(\mathbb{R}), \mathrm{Sp}_{2n}(\mathbb{R})$, where K_G is the maximal compact subgroup of $G = \mathrm{SO}_n(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C})$. Can one decide in finite time whether it is possible to lift an element $A_i \in \overline{A_i}$ from every conjugacy class such that $\prod A_i = \mathrm{Id}$? We first explain the case of the $\overline{A_i}$ having eigenvalues of finite order in which case this problem *algebrizes* as follows. Let $p : Y \rightarrow X$ be a Galois cover of smooth projective curves ramified at $\{x_i\}_{i \leq m}$ with ramification indices $\{n_i\}_{i \leq m}$. Let \tilde{Y} be the simply conneted cover of Y . Then the deck transformation group Γ of $\tilde{Y} \rightarrow X$ identifies with the free group on $2g(X) + m$ generators $\{A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_m\}$ quotiented by the relations $\prod [A_i, B_i] C_j = \mathrm{Id}$ and $C_i^{n_i} = \mathrm{Id}$. One knows that conjugacy classes of representations (resp. irreducible representations) of $\Gamma \rightarrow K_G$ bijectively correspond to polystable (resp. stable) π - G bundles on Y where π is the Galois group of $p : Y \rightarrow X$ (cf. [1]). If we take $X = \mathbb{P}^1$, then since $g_X = 0$ so the representations of Γ correspond to just picking an ordered set of elements A_i of K_G of order n_i multiplying to one. Fixing the conjugacy class of C_j in K_G is equivalent to fixing the parabolic datum in Definitions 2.0.3, 3.0.21 and 4.0.4. Thus the question of whether it is possible to pick elements A_i (resp. from irreducible representation) becomes equivalent to the algebraic problem of existence of polystable (resp. stable) parahoric G -bundles on \mathbb{P}^1 . For the more general case of $\overline{A_i}$ having elements of arbitrary order, it is possible to give explicit bounds, such that there exists a solution to the given $\overline{A_i}$ if

and only if there exists a solution for slightly perturbed \overline{A}_i which have all elements of finite order.

We now state the conditions in terms of inequalities. Let \tilde{C}_i be conjugacy classes of SO_n (resp. Sp_{2n}) in the following form: $(\exp(2\pi i\lambda_1) \leq \dots \leq \exp(2\pi i\lambda_n))$ where λ_i are real numbers such that $\lambda_i + \lambda_{n+1-i} = 0$ and $\lambda_n \leq 1/2$. We shall call this form standard and denote by $\tilde{\alpha}_j = (\lambda_1^j, \dots, \lambda_n^j)$ for the conjugacy class \tilde{C}_j .

We quickly recall the definition of Gromov–Witten numbers. Let R denotes the set of parabolic points. For $w \in R$, we consider generic complete orthogonal grassmanian G_w^\bullet on \tilde{V}_w . For a subset $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$, define the Schubert variety $\Omega_I^O(G^\bullet) = \{L \in Gr(r, \tilde{V}_w) \mid \dim(L \cap G^{i_j}) \geq j \text{ for all } 1 \leq j \leq r\}$. Let W be a vector bundle on \mathbb{P}^1 such that $W^* \simeq W$ (for our purposes it will be either the trivial bundle or $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{n-2}$). Define $Gr(d, r, W)$ to be the moduli space of isotropic sub-bundle of W of rank r and degree d . For $p \in \mathbb{P}^1$, define projection maps $\pi_p : Gr(d, r, W) \rightarrow Gr(r, W_p)$ to the fiber of W at p . We call a Schubert State $\mathfrak{J} = (d, r, W, \{I_w\}_{w \in R})$ where $I_w \subset \{1, \dots, n\}$ of cardinality r and d is an integer. For a Schubert state \mathfrak{J} define $\langle \mathfrak{J} \rangle$ to be the number of points in the intersection $\Omega^O(\mathfrak{J}, W, G^\bullet) = \cap_{w \in R} \pi_w^{-1}[\Omega_{I_w}^O(G_w^\bullet)] \subset Gr(d, r, W)$. We can now state our main theorems:

Theorem 1.0.1. Let $\{\tilde{C}_w\}_{w \in R}$ be conjugacy classes of SO_n in the standard form. Then it is possible to pick elements $C_w \in \tilde{C}_w$ such that $\prod C_w = \text{Id}$ if and only if either of the following conditions holds

- (1) given any $1 \leq r \leq n/2$ and any choice of subsets $\{I_w\}_{w \in R}$ of cardinality r of $\{1, \dots, n\}$, whenever $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_d = 1$ then $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d \leq 0$.
- (2) Let $W = \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{n-2}$. For every Schubert State $\mathfrak{J} = (d, r, W, \{I_w\}_{w \in R})$, whenever $\langle \mathfrak{J} \rangle = 1$, then for $I_w \in \mathfrak{J}$, we should have $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d \leq 0$.

It is possible to pick elements $\{C_w\}$ forming an irreducible set if and only if

- (1) whenever $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_d \neq 0$ or is ∞ then $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d < 0$.
- (2) whenever $\langle \mathfrak{J} \rangle \neq 0$ or is ∞ , then for $I_w \in \mathfrak{J}$, we should have $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d < 0$.

We recall that a subset H of a maximal compact subgroup K of a group G is said to be *irreducible* if $\{Y \in \mathfrak{g} \mid \text{adh}(Y) = Y, \forall h \in H\} = Z(\mathfrak{g})$.

Theorem 1.0.2. Let $\{\tilde{C}_w\}_{w \in R}$ be conjugacy classes of Sp_{2n} in the standard form. Then it is possible to pick elements $C_w \in \tilde{C}_w$ such that $\prod C_w = \mathrm{Id}$ if and only if given any $1 \leq r \leq n/2$ and any choice of subsets $\{I_w\}_{w \in R}$ of cardinality r of $\{1, \dots, n\}$, whenever $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_d = 1$ then $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d(\leq) 0$. The elements $\{C_w\}$ form an irreducible set if and only if whenever $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_d \neq 0$ or is ∞ then $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d < 0$.

By Proposition 5.5.4 we reduce the above question for the Spin group to the group SO_n (cf. Remark 5.5.5). Given conjugacy classes for Spin_n , the above questions are answered affirmatively for Spin_n if and only if for the associated conjugacy classes of SO_n , they are answered affirmatively.

1.1. Remarks on Notation. We have followed the system of notation in [18, CSS]. So a Galois cover of smooth projective groups $p : Y \rightarrow X$ has Galois group π and not Γ as in [1]. If V is a vector bundle on a curve X and x is a point of X , then we denote by V_x both the localization at x of the locally free sheaf of sections of V and the underlying vector space at x . This abuse will lighten the notations and from the context our meaning will be clear. Similarly for a quadratic form q on V , we denote both the localization and the evaluation at x by q_x and for a group scheme $G \rightarrow X$, both the closed special fiber and the stalk at x are denoted G_x . A quadratic form on a vector bundle \mathcal{F}^d (or its localization \mathcal{F}_w^d at w) is denoted by upper indices q^d but on vector spaces $q_{m/2} : G^{m/2} \rightarrow \mathbb{C}$ it is denoted by lower indices. We identify a vector bundle V with its sheaf of sections. Any statement with ‘(semi)-stability’ should be read as two statements, one with stability and the other with semi-stability. Similarly (\leq) should be read as \leq and $<$. We shall write SO_n instead of $\mathrm{SO}(n)$ and denote pull-backs $p^*(V)$ by p^*V , again to lighten notation.

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2. DEGENERATE ORTHOGONAL BUNDLES WITH FLAGS

We begin by describing our setup. Let G be an affine algebraic group together with a representation $G \rightarrow \mathrm{GL}(W)$. This section is devoted to the case of the standard representation $O_n \rightarrow \mathrm{GL}(W)$. Let $p : Y \rightarrow X$ be a Galois cover of smooth projective curves with Galois group π . By a π - G bundle we mean a principal G -bundle E on Y to which the action of π lifts. The theme of this section is to describe the additional data on the parabolic bundle $E(W)$ that captures the information that $E(W)$ is obtained as an extension of structure group from E .

Remark 2.0.1. The following definition generalizes the notion of *degenerate symplectic (resp. orthogonal) bundle* in [5] (cf also Remarks 2.0.8 and 5.1.3 for (semi)-stability).

Definition 2.0.2. We say that a quadratic bundle $q : V \rightarrow V^* \otimes L$ with values in a line bundle L has singularities of order $\leq r$ if for $S = (V^* \otimes L)/q(V)$ the following natural map is injective

$$S \rightarrow \bigoplus_{x \in \mathrm{Supp}(S)} S_x \otimes (\bigoplus_{i \leq r} \mathcal{O}_x / m_x^i).$$

Definition 2.0.3. A *degenerate orthogonal bundle with flags* (V, q, F^\bullet, L) is a vector bundle V on X endowed with the datum

- (1) a quadratic form $q : V \rightarrow V^* \otimes L$ with singularities of order ≤ 1 at a finite subset R of points of X ,
- (2) a flag structure $\{0\} \subsetneq F_x^{m_x} \subsetneq F_x^{m_x-1} \subsetneq \dots \subsetneq F_x^1 \subsetneq F_x^0 = V_x$ for each point $x \in R$, where the number m_x can vary with $x \in R$,

satisfying the conditions

- (1) (compatibility of quadratic form q and the flags) for every $x \in R$ we have $F_x^1 = \mathrm{Ker}(q_x : V_x \rightarrow (V^* \otimes L)_x)$
- (2) we denote by $F^1(V)$ the vector bundle obtained by pulling back $0 \rightarrow V(-\sum_{x \in R} x) \rightarrow V \rightarrow \bigoplus_{x \in R} V_x \rightarrow 0$ by $\bigoplus_{x \in R} F_x^1 \hookrightarrow \bigoplus_{x \in R} V_x$. Then q restricted to $F^1(V)$ factorizes as q_1 through $L(-R)$:

$$\begin{array}{ccc} F^1(V) \otimes F^1(V) & \xrightarrow{q} & L \\ & \searrow q^1 & \uparrow \\ & & L(-R) \end{array}$$

- (3) the quadratic form q^1 induces a non-degenerate quadratic form

$$q_{1,x} : F_x^1 \rightarrow L(-R)_x \simeq \mathbb{C}.$$

- (4) for $i \geq (1 + m_x)/2$, the flags F_x^i are isotropic for $(F_x^1, q_{1,x})$ and the remaining are obtained by orthocomplementation.

Remark 2.0.4. Notice that on successive quotients $G_x^i = F_x^i/F_x^{i+1}$ we have perfect pairings $q_{i,x} : G_x^{m-i} \times G_x^i \rightarrow \mathbb{C}$ for $1 \leq i \leq m_x/2$ and on quotients $G_x^{(1+m_x)/2}$ (if m_x is odd) and G_x^0 we have non-degenerate quadratic forms $q_{(1+m_x)/2,x}$ and $q_{0,x} : V_x/F_x^1 \rightarrow L_x/m_x \simeq \mathbb{C}$. Notice also that for any x we have $\dim G_x^0 + \dim G_x^{(1+m_x)/2} \equiv \text{rank } V \pmod{2}$ if m_x is odd and $\dim G_x^0 \equiv \text{rank } V \pmod{2}$ if m_x is even. Conversely, it is clear that the existence of perfect pairings on successive quotients G_x^i for $i \geq 1$ endow F_x^1 with a quadratic form and non-degenerate quadratic form on G_x^0 endows, together with $(F_x^1, q_{1,x})$, a degenerate quadratic form at the stalk V_x of the desired type.

Remark 2.0.5. The compatibility of the global quadratic form q and the local ones can be expressed more explicitly as follows: if we choose a basis \mathfrak{B} of the localization of the free module V_x such that \mathfrak{B} induces a basis of the underlying vector space V_x in which q_i 's can be expressed in the standard anti-diagonal form, then in \mathfrak{B} the quadratic form q_x is expressed by $\begin{bmatrix} J_1 & \\ & tJ_2 \end{bmatrix}$, here t is the local parameter of L_x and J_1 and J_2 are the standard anti-diagonal matrices of sizes $\dim(V_x/F_x^1)$ and $\dim(F_x^1)$ respectively.

We have defined degenerate orthogonal bundles with values in a line bundle L because this is the right formulation to define Hecke-modifications by ‘isotropic subspaces’. We explain how to do this in [13]. The most important case is when $L = \mathcal{O}_X$, where we shall often abbreviate to (V, q, F^\bullet) . The following proposition shows that the case of a general L reduces to $L = \mathcal{O}_X$ by taking a square root of L , if need be by going to a cover.

Proposition 2.0.6. Assume that L is a line bundle of odd degree on X . Let $p : \tilde{X} \rightarrow X$ be a two-sheeted cover such that $p^*L = L_1^2$. Then given a degenerate orthogonal bundle with flags (V, q, F^\bullet, L) and a choice of square root L_1 , we can canonically associate to its pullback to \tilde{X} a degenerate orthogonal bundle with values in the trivial bundle $\mathcal{O}_{\tilde{X}}$.

Proof. Consider the sequence $0 \rightarrow V \rightarrow^q V^* \rightarrow S \rightarrow 0$. The support of S lies in the set of parabolic points R . For $x \in R$, by tensorization with the skyscraper sheaf \mathbb{C}_x , we have a canonical identification between F_x^1 and $\text{Tor}_{\mathcal{O}_X}^1(S, \mathbb{C}_x)$ since the depth of S is one. Pulling back to \tilde{X} and tensoring with $\mathbb{C}_{\tilde{x}}$ for \tilde{x} lying over x , we obtain

$$(2.0.1) \quad 0 \rightarrow p^*V \otimes L_1^{-1} \xrightarrow{p^{*q}} p^*V^* \otimes L_1 \rightarrow p^*S \otimes L_1^{-1} \rightarrow 0,$$

which again, owing to the fact that depth of $p^*S \otimes L_1^{-1}$ is one, implies that $\ker(p^*q)_{\tilde{x}} = \text{Tor}_{\mathcal{O}_{\tilde{X}}}^1(p^*S \otimes L_1^{-1}, \mathbb{C}_{\tilde{x}})$. As the operations of taking geometric fiber and pull-back commute, we have canonical isomorphisms $\text{Tor}_{\mathcal{O}_{\tilde{X}}}^1(p^*S, \mathbb{C}_{\tilde{x}}) \simeq p^*\text{Tor}_{\mathcal{O}_X}^1(S, \mathbb{C}_x)_{\tilde{x}}$. We have thus have canonical isomorphisms $\text{Tor}_{\mathcal{O}_{\tilde{X}}}^1(p^*S \otimes L_1^{-1}, \mathbb{C}_{\tilde{x}}) \simeq \text{Tor}_{\mathcal{O}_{\tilde{X}}}^1(p^*S, \mathbb{C}_{\tilde{x}}) \otimes L_1^{-1} \simeq p^*\text{Tor}_{\mathcal{O}_X}^1(S, \mathbb{C}_x)_{\tilde{x}} \otimes L_1^{-1}$. By writing L_1 as tensor product of line bundles or their inverses which admit sections, we obtain an identification, well defined upto scalars, between $\ker(p^*q)_{\tilde{x}} \simeq \text{Tor}_{\mathcal{O}_{\tilde{X}}}^1(p^*S \otimes L_1^{-1}, \mathbb{C}_{\tilde{x}})$ and $\text{Tor}_{\mathcal{O}_X}^1(S, \mathbb{C}_x) \simeq F^1(V)_x$. This allows us to transport the entire flag structure $\{F_x^i\}$ at x to \tilde{x} . We can thus define $V_1 = p^*V \otimes L_1^{-1}$, and $F_{\tilde{x}}^1(V) = \text{Tor}_{\mathcal{O}_{\tilde{X}}}^1(p^*S \otimes L_1^{-1}, \mathbb{C}_{\tilde{x}})$ for \tilde{x} lying over R and associate $(V_1, p^*q, \{F_{\tilde{x}}^\bullet\}_{p(\tilde{x}) \in R}, \mathcal{O}_{\tilde{X}})$ to (V, q, F^\bullet, L) . The compatibility conditions of Definition 2.0.3 now follow: (1) and (4) by definition of F_x^1 and (2) and (3) by Remark 2.0.5. \square

If the degree of L is even, then it is a square on X . The proof of Proposition 2.0.6 shows how one can associate to (V, q, F^\bullet, L) a bundle with values in the trivial bundle.

Definition 2.0.7. A parabolic degenerate orthogonal bundle with flags is a degenerate orthogonal bundle with flag together with a sequence of rational numbers $0 \leq \alpha_x^1 < \dots < \alpha_x^i < \dots < \alpha_x^{m_x} < 1$ associated to the subspaces F_x^i . They are increasing with the associated vector space becoming smaller.

Remark 2.0.8. In the context of [5], there is only one flag namely $\ker(q_x) \subset V_x$ at each parabolic point with weight $1/2$.

For a ramification point $y \in \text{ram}(p)$ let π_y denote the isotropy subgroup at y , which is well known to be cyclic. The action of π_y , provides a canonical decomposition

$$W_y = \oplus_{g \in \pi_y^*} W_{y,g}$$

in terms of the character group π_y^* of π_y , where $W_{y,g} = \{w \in W_y \mid \gamma w = g(\gamma)w \forall \gamma \in \pi_y\}$. Let B_y denote the bilinear form at y . For $g_1, g_2 \in \pi_y^*$, if $g_1 \neq g_2^{-1}$ then $W_{g_1} \perp W_{g_2}$ under B_y , otherwise there is a perfect pairing $B_{y,g} : W_g \times W_{g^{-1}} \rightarrow \mathbb{C}$. The isotropy subgroup π_y acts on the cotangent space m_y/m_y^2 at y through a representation which defines a generator h_y of π_y^* . For any $g \in \pi_y$ let i_g denote the natural number defined by $h^{i_g} = g$. Let $W_{|\pi|}$ denote $p^*p_*^*W$.

Proposition 2.0.9. The quadratic form q' on W goes down to V as a quadratic form q of singularity ≤ 1 .

Proof. We have $i_g + i_{g-1} = |\pi_y^*|$ for $g \neq \{e\}$. Now the bundle $W_{|\pi|}$ can also be described by the Hecke modification

$$(2.0.2) \quad 0 \rightarrow W_{|\pi|} \rightarrow W \rightarrow \bigoplus_{y \in \text{ram}(p)} \bigoplus_{g \in \pi_y^* \setminus \{e\}} W_{y,g} \otimes_{\mathbb{C}} \mathcal{O}_y / m_y^{|\pi_y^*| - i_g} \rightarrow 0$$

Since $|\pi_y^*| - i_g + |\pi_y^*| - i_{g-1} = |\pi_y^*|$, so the restriction of q' to $W_{|\pi|}$ has singularities of order $|\pi_y^*|$ at y and q has thus singularities of order 1 at x . \square

The following Proposition is readily checked.

Proposition 2.0.10. For y_1 and y_2 in the same fiber of p , there exists the following canonical isomorphisms

- (1) $\alpha : \pi_{y_2}^* \rightarrow \pi_{y_1}^*$ mapping h_{y_2} to h_{y_1} . It can be obtained by conjugation by any $\theta \in \pi$ satisfying $\theta(y_1) = y_2$.
- (2) $\beta : \mathbb{P}(W_{y_1}) \rightarrow \mathbb{P}(W_{y_2})$ which restricts to canonical isomorphisms $\beta_g : \mathbb{P}(W_{y_1,g}) \rightarrow \mathbb{P}(W_{y_2,\alpha^*(g)})$ for any $g \in \pi_{y_1}^*$.
- (3) between the bilinear forms B_{y_1} and B_{y_2} i.e the following diagram commutes

$$\begin{array}{ccc} \mathbb{P}(W_{y_1}) & \xrightarrow{q'_{y_1}} & \mathbb{P}(W_{y_1}^*) \\ \downarrow \beta & & \uparrow \beta^* \\ \mathbb{P}(W_{y_2}) & \xrightarrow{q'_{y_2}} & \mathbb{P}(W_{y_2}^*) \end{array}$$

In particular, the identification is independent of θ mapping y_1 to y_2 .

Let us recall the Rees lemma in homological algebra. In this paper, we will often use it for the case $n = 0$ to make an extension of a skyscraper sheaf by a vector bundle.

Theorem 2.0.11 (Rees). Let R be a ring and $x \in R$ be an element which is neither a unit nor a zero divisor. Let $R^* = R/(x)$. For an R -module M , suppose moreover that x is regular on M . Then there is an isomorphism

$$\text{Ext}_{R^*}^n(L^*, M/xM) \simeq \text{Ext}_R^{n+1}(L^*, M)$$

for every R^* -module L^* and every $n \geq 0$.

Proposition 2.0.12. Let (V, q) be a quadratic bundle on X such that q is generically an isomorphism. Then putting $S = V^*/q(V)$, there is a natural isomorphism of the skyscraper sheaves $S \simeq \text{Ext}_X^1(S, \mathcal{O}_X)$.

Proof. This follows immediately by applying the functor $\text{Hom}_X(-, \mathcal{O}_X)$ to the short exact sequence $0 \rightarrow V \xrightarrow{q} V^* \rightarrow S \rightarrow 0$ and remarking that $q^* = q$. \square

Remark 2.0.13. The following theorem generalizes to arbitrary Galois cover of curves, the [5, Prop 1.3] which is for two-sheeted covers.

Theorem 2.0.14. Let W be a π - GL_n bundle on Y such that $W(\mathrm{GL}_n/O_n) \rightarrow Y$ admits a π -invariant section q' . Then to such a bundle we can canonically associate a parabolic degenerate orthogonal bundle $(V, q, F^\bullet, \mathcal{O}_X)$ with flags at precisely the branch points of $p : Y \rightarrow X$ and parabolic weights symmetric about $1/2$. Conversely, let $(V, q, F^\bullet, \mathcal{O}_X)$ be a parabolic orthogonal bundle on a smooth projective curve X with weights symmetric about $1/2$, then there exists a Galois cover $p : Y \rightarrow X$ with Galois group π along with a vector bundle W on Y and a π -invariant section $q' : Y \rightarrow W(\mathrm{GL}_n/O_n)$ such that (W, q') is mapped to (V, q) by the first part of the theorem.

Proof. By Proposition 2.0.10 part (1) and (2), it suffices to treat the case of one ramification point $y \in Y$. Tensoring (2.0.2) with $\mathcal{O}_y/m_y^{|\pi_y|}$, for

$$S = \bigoplus_{g \in \pi_y^* \setminus \{e\}} W_{y,g} \otimes_{\mathbb{C}} \mathcal{O}_y/m_y^{|\pi_y^*| - i_g}$$

we get a $\mathcal{O}_y/m_y^{|\pi_y^*|}$ -submodule

$$(2.0.3) \quad S \simeq \mathrm{Tor}_1(S, \mathcal{O}_y/m_y^{|\pi_y^*|}) \hookrightarrow W_{|\pi|,y}/m_y^{|\pi_y^*|}.$$

Let t denote the local parameter of m_y . For $1 \leq i \leq |\pi_y|$, the image of S under the composition of the endomorphism $W_{|\pi|,y}/m_y^{|\pi|} \xrightarrow{\mathrm{mult}(t_y^{i-1})} W_{|\pi|,y}/m_y^{|\pi|}$ followed by the projection $W_{|\pi|,y}/m_y^{|\pi|} \rightarrow W_{|\pi|,y}/m_y^{|\pi|-1}/m_y^{|\pi|} \simeq W_{|\pi|,y}$ defines subspaces $F_y^i \subset W_{|\pi|,y}$. This sequence of subspaces are naturally filtered $F_y^{|\pi_y|-1} \subset \cdots \subset F_y^1 \subset W_{|\pi|,y}$ owing to the fact that S is an $\mathcal{O}_y/m_y^{|\pi_y|}$ -submodule of $W_{|\pi|,y}/m_y^{|\pi|}$. Some inclusions may be equalities, so we extract a reduced filtration keeping only distinct subspaces by associating the rational number $\alpha_y^i = i/|\pi_y^*|$ to F_y^i if $F_y^i \neq F_y^{i-1}$. This defines a weighted filtration of V_x for $x = p(y)$, which we denote by F_x^i . By (2.0.3), we also obtain that $W_{y,h^i} = F_y^i/F_y^{i+1}$. The perfect pairing between $W_{y,g}$ and $W_{y,g^{-1}}$ goes down to X by Proposition 2.0.10 part (3) to give a perfect pairing between G_x^i and $G_x^{|\pi_y|-i}$ where $G_x^i = F_x^i/F_x^{i+1}$. When $g = g^{-1}$ ie for $i_g = 0$ and $i_g = (1+|\pi_y|)/2$ we get a non-degenerate quadratic form on $G^0 = V_x/F^1$ and $G^{(1+|\pi_y|)/2}$. On the other hand, tensoring (2.0.2) with \mathbb{C}_y we see that $F_y^1 = \mathrm{Tor}_1(\bigoplus_{g \in \pi_y^* \setminus \{e\}} W_{y,g}, \mathbb{C}_y) \simeq \bigoplus_{g \in \pi_y^* \setminus \{e\}} W_{y,g}$ becomes a subspace of $W_{|\pi|,y} \simeq V_x$ and is isomorphic to F_x^1 . Also the subspaces $\bigoplus_{g \in \pi_y^* \setminus \{e\}} W_{y,g}$ and $W_{y,e}$ are perpendicular to each other. So the restriction of the quadratic form to F_y^1 descends

to F_x^1 as $q_{1,x}$. Now the compatibility conditions (3) and (4) of Definition 2.0.3 follow by Remark 2.0.5 and Remark 2.0.4. The restriction of the quadratic from q' to $W_{|\pi|}$ becomes degenerate at the fiber at y , but induces a non-degenerate form on $W_{|\pi|,y}/F_y^1$. This descends to the condition (1) of Definition 2.0.3.

Condition (2) that the order of degeneracy of q on $F^1(V)$ is only one follows from Prop 2.0.9.

This completes the proof of one direction in Theorem 2.0.14.

Lemma 2.0.15. Let r be an integer, V a vector bundle and $R \subset X$ a finite set of points. Let rR denote the divisor $\sum_{x \in R} rx$. There exists a canonical extension

$$(2.0.4) \quad 0 \rightarrow V \rightarrow V(rR) \rightarrow V \otimes \bigoplus_{x \in R} \mathcal{O}_x / m_x^r \rightarrow 0$$

which is universal in the sense that any extension of a skyscraper sheaf S of depth less than r with support in R can be obtained by (2.0.4) by pull-back by a homomorphism $S \rightarrow V \otimes \bigoplus_{x \in R} \mathcal{O}_x / m_x^r$.

Proof. We denote $V^*(-rR)$ as the $\text{Ker}(V^* \rightarrow V^* \otimes \bigoplus_{x \in R} \mathcal{O}_x / m_x^r)$. Then dualizing $0 \rightarrow V^*(-rR) \rightarrow V^* \rightarrow V^* \otimes \bigoplus_{x \in R} \mathcal{O}_x / m_x^r \rightarrow 0$ we get

$$0 \rightarrow V \rightarrow V(rR) \rightarrow \text{Ext}^1(\bigoplus_{x \in R} V_x^* / m_x^r, \mathcal{O}_X) \rightarrow 0.$$

By Rees's Theorem 2.0.11 we have

$$\text{Ext}_X^1(\bigoplus_{x \in R} V_x^* / m_x^r, \mathcal{O}_X) = \bigoplus_{x \in R} \text{Hom}_{\mathcal{O}_x / m_x^r}(V_x^* / m_x^r, \mathcal{O}_x / m_x^r) = \bigoplus_{x \in R} V_x / m_x^r.$$

So the sequence becomes $0 \rightarrow V \rightarrow V(rR) \rightarrow \bigoplus_{x \in R} V_x / m_x^r \rightarrow 0$. On the other hand by Rees's Theorem again, we have $\text{Ext}_X^1(S, V) = \text{Hom}_{\bigoplus_{x \in R} \mathcal{O}_x / m_x^r}(S, \bigoplus_{x \in R} V_x / m_x^r)$. More explicitly, this corresponds to taking the pull-out by $\phi : S \rightarrow \bigoplus_{x \in R} V_x / m_x^r$ of (2.0.4) to get an extension and from an extension $0 \rightarrow V \rightarrow W \rightarrow S \rightarrow 0$, by taking tensor product with $\bigoplus_{x \in R} \mathcal{O}_x / m_x^r$, we get an injective homomorphism $S = \text{Tor}_1(S, \bigoplus_{x \in R} \mathcal{O}_x / m_x^r) \rightarrow V \otimes \bigoplus_{x \in R} \mathcal{O}_x / m_x^r$. \square

Let $p : Y \rightarrow X$ realise a Galois cover such that for every point $x \in R$ the ramification index $r_x = |\pi_y^*|$ (for $p(y) = x$) is a multiple of the least common divisor l_x of the denominators of the weights α_x^i of the Flag at x . The existence of such a cover is classically well known.

Let S denote $V^*/q(V)$. Taking the pull-back of $0 \rightarrow V \xrightarrow{q} V^* \rightarrow S \rightarrow 0$ to $p : Y \rightarrow X$, we get

$$(2.0.5) \quad 0 \rightarrow p^*V \xrightarrow{p^*q} p^*V^* \rightarrow p^*S \rightarrow 0.$$

Define a sequence of flags $F_y^j \subset p^*V_y \simeq V_x$ for $1 \leq j \leq r_x$ as $F_y^j = F_x^i$ whenever $\alpha_x^{i-1}r_x < j \leq \alpha_x^i r_x$. In particular, $F_y^{r_x} = \{0\}$. Thus for all $y \in p^{-1}(R)$ we continue to have a perfect pairing $G_y^i \times G_y^{r_x-i} \rightarrow \mathbb{C}$.

Then define a $\mathcal{O}_y/m_y^{|\pi_y^*|}$ -submodule T of $p^*V_y/m_y^{|\pi_y^*|}$ as the sub-module generated by $F_y^i \otimes_{\mathbb{C}} m_y^{r_x-i}/m_y^{|\pi_y^*|}$ for $1 \leq i \leq r_x$. Owing to the inclusions $F_y^{r_x} \subset \dots \subset F_y^1 \subset p^*V_y$, T is simply $\sum_{1 \leq i \leq r_x} F_y^i \otimes_{\mathbb{C}} m_y^{r_x-i}/m_y^{r_x}$. It can be expressed as a vector space as follows

$$T = \oplus_{1 \leq i \leq r_x} F_y^i \otimes_{\mathbb{C}} m_y^{r_x-i}/m_y^{r_x-i+1}.$$

We take pull-out of (2.0.5) by $T \rightarrow p^*S$ to get

$$(2.0.6) \quad 0 \rightarrow p^*V \rightarrow W \rightarrow T \rightarrow 0.$$

The sequence (2.0.6) on dualizing gives

$$(2.0.7) \quad 0 \rightarrow W^* \rightarrow p^*V^* \rightarrow \text{Ext}^1(T, \mathcal{O}_Y) \rightarrow 0$$

We firstly claim that the composite of $p^*V \xrightarrow{p^*q} p^*V^* \rightarrow \text{Ext}^1(T, \mathcal{O}_Y)$ is zero. This would show that p^*q factors through $W \rightarrow p^*V^*$.

Firstly the sequence (2.0.5) belongs to $\text{Ext}^1(p^*S, p^*V)$ and arises as the image of $\text{Id} \in \text{Hom}(p^*S, p^*S)$. By the commuting squares

$$\begin{array}{ccccccc} \longrightarrow & \text{Hom}(p^*S, p^*S) & \longrightarrow & \text{Ext}^1(p^*S, p^*V) & \longrightarrow & \text{Ext}^1(p^*S, p^*V^*) & \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \text{Hom}(T, p^*S) & \longrightarrow & \text{Ext}^1(T, p^*V) & \longrightarrow & \text{Ext}^1(T, p^*V^*) & \end{array}$$

the push-out of (2.0.6) by $p^*V \rightarrow p^*V^*$ is the zero extension

$$(2.0.8) \quad \begin{array}{ccccccc} 0 & \longrightarrow & p^*V & \longrightarrow & W & \longrightarrow & T \longrightarrow 0 \\ & & \downarrow p^*q & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p^*V^* & \longrightarrow & p^*V^* \oplus T & \longrightarrow & T \longrightarrow 0. \end{array}$$

Now viewing $v \in p^*V$ as a form on p^*V^* , we see that the composite of $v \circ p^*q = p^*(q)(v)$. So since the bottom row of (2.0.8) is split, so the push-outs by $p^*V^* \xrightarrow{v} \mathcal{O}_X$ are split. Thus the push-out of (2.0.6) by $p^*V \xrightarrow{p^*q(v)} \mathcal{O}_Y$ is split. This shows that we have a factorization (first q^1 and then $p^*S \rightarrow \text{Ext}^1(T, \mathcal{O}_Y)$)

$$(2.0.9) \quad \begin{array}{ccccccc} & & p^*V & & & & \\ & \swarrow q^1 & \downarrow p^*q & & & & \\ 0 & \longrightarrow & W^* & \longrightarrow & p^*V^* & \longrightarrow & \text{Ext}^1(T, \mathcal{O}_Y) \longrightarrow 0 \\ & & & & \downarrow & \nearrow & \\ & & & & p^*S & & \end{array}$$

Let Q denote the $\mathcal{O}_y/m_y^{|\pi_y^*|}$ -module which is quotient of $T \rightarrow p^*S$. It can be expressed as

and we have $0 \rightarrow \mathrm{Ext}_Y^1(Q, \mathcal{O}_Y) \rightarrow \mathrm{Ext}_Y^1(p^*S, \mathcal{O}_Y) \rightarrow \mathrm{Ext}_Y^1(T, \mathcal{O}_Y) \rightarrow 0$. So by Proposition 2.0.12 and diagram (2.0.9), we may replace p^*S by $\mathrm{Ext}_Y^1(p^*S, \mathcal{O}_Y)$ in 2.0.9 to obtain

Now rearranging terms we have

By Rees's Theorem 2.0.11, we have

Since for $i \geq 1$ we have a canonical isomorphism of $\mathcal{O}_y/m_y^{|\pi_y^*|}$ -modules

so rearranging terms again by (2.0.10) we obtain $\mathrm{Ext}^1(Q, \mathcal{O}_Y) = T$. Thus we have $0 \rightarrow p^*V \xrightarrow{q_1} W^* \rightarrow T$. This sequence on dualizing gives $0 \rightarrow W \xrightarrow{q_1^*} p^*V^* \rightarrow \mathrm{Ext}^1(T, \mathcal{O}_Y) \rightarrow 0$.

$$(2.0.11) \quad \begin{array}{ccccccc} & & & W & & & \\ & & & \downarrow q_1^* & & & \\ & & q' \swarrow & & \searrow & & \\ 0 & \longrightarrow & W^* & \longrightarrow & p^*V^* & \longrightarrow & \mathrm{Ext}^1(T, \mathcal{O}_Y) \longrightarrow 0. \end{array}$$

Corollary 2.0.16. The underlying degree of a parabolic degenerate orthogonal bundle with flags $(V, q, F_x^\bullet, \mathcal{O}_X)$ associated to a π - O_n bundle is equal to $-\frac{1}{2} \sum_{x \in R} \dim(F_x^1)$.

13

In [1, Theorem 2.4.1, version 1], it is proved that as one varies over the coverings $p : Y \rightarrow X$, the germ of local automorphism $\text{Aut}_{\pi-G}(E)_y$ realises *all* parahoric subgroups of $G(K_x)$ where K_x is the quotient field of \mathcal{O}_x for $x = p(y)$. So in Proposition 2.0.17, we describe this group scheme as automorphisms of parahoric orthogonal bundles.

Proposition 2.0.17. For $y \in Y$ a ramification point, the unit group $\text{Aut}_y^{\pi_y}(W, q') = O_n(\mathcal{O}_y)^{\pi_y}$ consisting of germs of local automorphisms identifies canonically with $\text{Aut}_x(V, q, F^\bullet)$ which consists of elements of $\text{GL}_n(\mathcal{O}_x)$ that preserve q_x and on the special fiber preserve the flag $F_x^{m_x} \subset \cdots \subset F_x^1 \subset V_x$ and respect the perfect pairings $q_{i,x} : G_x^i \times G_x^{m_x-i} \rightarrow \mathbb{C}$ and the quadratic forms $q_{0,x} : G_x^0 \rightarrow \mathbb{C}$ and $q_{(1+m_x)/2,x} : G_x^{(1+m_x)/2} \rightarrow \mathbb{C}$.

Proof. We use the notation in the proof of Theorem 2.0.14. Firstly for $g \in \pi_y^*$ the unit group $O_n(\mathcal{O}_y)^{\pi_y}$ preserves the g -eigenspaces $(W_y/m_y^i)_g \subset W_y/m_y^i$ for $i \geq 1$. By the following equality of skyscraper as sheaves of \mathcal{O}_Y -modules

$$S = \bigoplus_{g \in \pi_y^* \setminus \{e\}} W_{y,g} \otimes_{\mathbb{C}} \mathcal{O}_y/m_y^{|\pi_y^*|-i_g} = \bigoplus_{1 \leq i \leq |\pi_y^*|-1} (W_y/m_y^i)_{h^i}.$$

we deduce that for any $\theta \in O_n(\mathcal{O}_y)^{\pi_y}$, its action restricts well to S and that the projection $W_y \rightarrow S$ is $O_n(\mathcal{O}_y)^{\pi_y}$ -equivariant. So any $\theta \in O_n(\mathcal{O}_y)^{\pi_y}$ restricts well to the kernel $W_{|\pi|}$ of the projection to define its automorphism. Thus tensoring by $\mathcal{O}_y/m_y^{|\pi_y^*|}$ we see that any θ also defines an automorphism of $\text{Tor}_1(S, \mathcal{O}_y/m_y^{|\pi_y^*|}) \simeq S$. Now since the subspaces F_x^i have been defined as the image of S under multiplication by t_y^{i-1} on $W_{|\pi|,y}/m_y^{|\pi_y^*|}$ followed by projection onto $W_{|\pi|,y}m_y^{|\pi_y^*|-1}/m_y^{|\pi_y^*|} \simeq W_{|\pi|,y} \simeq V_x$, so any θ preserves this subspace too. Since such an automorphism is moreover π_y -invariant, so it goes down to give an automorphism of V . It is a formal check that this automorphism of V preserves q . The unit group also preserves the perfect pairing between $W_{y,g} \times W_{y,g^{-1}}$ so it preserves the pairing between G_y^i and $G_y^{r_x-i}$ and thus the pairing between G_x^i and $G_x^{m_x-i}$. Conversely, any element of $\theta \in \text{Aut}_x(V, q)$ that preserves the subspaces F_x^i also respects the inclusion

$$T = \bigoplus_{x \in R} \bigoplus_{1 \leq i \leq r_x} F_y^i \otimes_{\mathbb{C}} m_y^{r_x-i}/m_y^{r_x-i+1} \hookrightarrow p^*(V)_y/m_y^{|\pi_y^*|},$$

owing to the symmetry of the parabolic weights α_x^i about $1/2$ (this symmetry moreover forces that θ also respect the pairing). Therefore by the functoriality of Rees Lemma defines an automorphism of the short exact sequence

$$0 \rightarrow p^*V \rightarrow W \rightarrow T \rightarrow 0$$

and thus defines an element in $\text{Aut}_y^{\pi_y}(W)$. Since the left, the right and the extreme most squares commute in

$$\begin{array}{ccccccc} \pi^*V_y & \longrightarrow & W & \xrightarrow{q'} & W^* & \longrightarrow & \pi^*V_y^* \\ \downarrow & & \uparrow & & \downarrow & & \uparrow \\ \pi^*V_y & \longrightarrow & W & \xrightarrow{q'} & W^* & \longrightarrow & \pi^*V_y^* \end{array}$$

thus so does the one in the middle, this shows that $\theta \in \text{Aut}_y^{\pi_y}(W, q')$. \square

We define the group scheme associated to a degenerate orthogonal bundle with flags.

Definition 2.0.18. For a degenerate orthogonal bundle with flags $(V, q, F^\bullet, \mathcal{O}_X)$ as in Definition 2.0.3, we shall denote by $O(V, q, F) = O_q \rightarrow X$ its groups of automorphisms which is defined as the subgroup scheme of the twisted group scheme $\text{Aut}(V) \rightarrow X$ whose sections over an open subset U are defined as follows

$$O_q(U) = \{s \in \text{Aut}(V)(U) \mid s^*qs = q, \quad s(F) = F, \quad s \text{ preserves perfect pairings}\}.$$

Now a degenerate orthogonal bundle with flags $(V, q, F^\bullet, \mathcal{O}_X)$ can also be viewed as a torsor under the group scheme $O_q \rightarrow X$.

3. DEGENERATE SPECIAL ORTHOGONAL BUNDLES WITH FLAGS

By a π -special orthogonal structure on a π - O_n -bundle E we mean the following: denote $\{m_g\}_{g \in \pi}$ the linearization on E , we demand that there exists moreover a section s that makes the following diagram commute:

$$\begin{array}{ccccc} E & \xrightarrow{m_g} & E & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & E(O_n/\text{SO}_n) & \xrightarrow{\overline{m}_g} & E(O_n/\text{SO}_n) & \\ \downarrow & \nearrow s_Y & \downarrow & \nearrow s_Y & \\ Y & \xrightarrow{g} & Y & & \end{array}$$

In particular the bundle $E(V)$ needn't have trivial determinant, it can be a line bundle of order two. When it moreover has trivial determinant, we shall call it a π - SO_n -bundle. One can describe π -special orthogonal structures on X as follows:

Recall the group scheme $O_q \rightarrow X$ of Definition 2.0.18. Recall also that $\text{Aut}(V) \rightarrow X$ is equal to the twist ${}_V\text{GL}_{n,X}$ of the constant group

scheme $\mathrm{GL}_{n,X} \rightarrow X$ by V . One similarly uses the twist ${}_V\mathrm{SL}_{n,X}$ to define the group scheme SO_q .

Definition 3.0.19. The group scheme $\mathrm{SO}_q \rightarrow X$ is defined to be the fibered product of $O_q \rightarrow X$ with ${}_V\mathrm{SL}_{n,X} \rightarrow X$. Its local sections over an open subset U are given as follows

$$\mathrm{SO}(U) = \{s \in {}_V\mathrm{SL}_{n,X} \mid s^*qs = q, \quad s(F) = F, \quad s \text{ preserves perfect pairings}\}$$

Theorem 3.0.20. Let (W, q', s_Y) be a π - SO_n bundle on Y . Then to W we can canonically associate a degenerate orthogonal bundle with flags $(V, q, F_\bullet^\bullet, \mathcal{O}_X)$ on X such that $\dim F_x^1$ is even for every $x \in R$ and the quotient space $V(O_q/\mathrm{SO}_q) \rightarrow X$ admits a global section s_X . Conversely given such a degenerate orthogonal bundle with flags $(V, q, F_\bullet^\bullet, \mathcal{O}_X)$ and a section s_X , we can construct a Galois cover $p : Y \rightarrow X$ and a π - SO_n bundle W on Y which is mapped to $(V, q, F_\bullet^\bullet, \mathcal{O}_X, s_X)$ by the first part of the theorem.

Proof. The only thing that needs to be checked is that $\dim F_x^1$ is even for every $x \in R$. One knows that π_y acts on the fiber of the special orthogonal bundle (W, q', s_Y) through a representation $\rho_y : \pi_y \rightarrow \mathrm{SO}_n$ (cf [7, Prop 1, page 06]). Since π_y is cyclic, so its image lands inside a maximal torus of SO_n . This implies that dimension F_x^1 is even, both in cases when n is even and odd. This follows because F_x^1 corresponds to $\bigoplus_{g \in \pi_y^* \setminus \{e\}} W_{y,g}$ which must be even dimensional owing to the description of the maximal torus in both even and odd cases. Let E be the orthogonal bundle underlying W . The π -equivariant section $s_Y : Y \rightarrow E(O_n/\mathrm{SO}_n)$ descends to the section $s_X : X \rightarrow V(O_q/\mathrm{SO}_q)$ and conversely the pull-back of s_X to Y furnishes s_Y . \square

We make the above theorem into a definition.

Definition 3.0.21. A parabolic degenerate special orthogonal bundle is a degenerate orthogonal bundle with flags $(V, q, F_\bullet^\bullet, \mathcal{O}_X)$ together with a global section $s_X : X \rightarrow V(O_q/\mathrm{SO}_q)$.

So a parahoric special orthogonal bundle can be viewed as a torsor under $\mathrm{SO}_q \rightarrow X$.

Remark 3.0.22. Notice that for every $x \in R$ we have $G_x^{(1+m_x)/2}$ (if m_x is odd) is even dimensional unlike in the orthogonal case (compare with Remarks 2.0.4 and 4.0.6). Thus $\dim G_x^0 \equiv \mathrm{rank} V(\mathrm{mod} 2)$ for every $x \in R$.

3.1. Interpretation of π - SO_n bundles as parahoric bundles.

Proposition 3.1.1. Let E be a π - SO_n bundle on $p : Y \rightarrow X$ with non-trivial Chern class i.e it doesn't admit a lift to a π - Spin_n bundle. Then there exists a Galois cover $p_1 : \tilde{Y} \rightarrow Y$, such that the pull-back of \tilde{E} admits a lift to a $\tilde{\pi}$ - Spin_n bundle, where $\tilde{\pi}$ is the Galois group of $\tilde{Y} \rightarrow X$.

Proof. Since SO_n is a *semi-simple* group, so it comes from a representation $\rho : \Gamma \rightarrow \mathrm{SO}_n$ where Γ is the deck transformation group of the simply connected cover $\mathbb{H}_Y \rightarrow Y$ over X . Let Γ be given as the quotient of the free abelian group on $\{a_i, b_i, c_j\}$ where $i \leq g_X$ and $j \leq |R|$ quotiented by the relation and $\prod [a_i, b_i] \prod c_j = 1$ and $c_j^{n_j} = 1$ where n_j is the ramification index at the j th branch point. Let $\tilde{\Gamma}$ be the free group on $\{\tilde{a}_i, \tilde{b}_i, \tilde{c}_j, d\}$ where $i \leq g_X$ and $j \leq |R|$ quotiented by the relations $\prod [\tilde{a}_i, \tilde{b}_i] \prod \tilde{c}_j d = 1$ and $\tilde{c}_j^{2n_j} = 1, d^2 = 1$. We lift ρ to $\tilde{\rho}$ by defining $\tilde{\rho}(\tilde{a}_i), \tilde{\rho}(\tilde{b}_i)$ and $\tilde{\rho}(\tilde{c}_j)$ to be any lifts to Spin_n of $\rho(a_i), \rho(b_i)$ and $\rho(c_j)$ respectively. Under the map $\mathrm{Spin}_n \rightarrow \mathrm{SO}_n$, the images of $\tilde{\rho}(\tilde{c}_j)^{n_j}$ and $\prod [\tilde{\rho}(\tilde{a}_i), \tilde{\rho}(\tilde{b}_i)] \prod \tilde{\rho}(\tilde{c}_j)$ is the neutral element, so they belong to the $\ker : \mathrm{Spin}_n \rightarrow \mathrm{SO}_n$. Now it is clear that $\tilde{\rho}(\tilde{c}_j)^{2n_j} = 1$ and one can define $\tilde{\rho}(d)$ such that the relation $\prod [\tilde{a}_i, \tilde{b}_i] \prod \tilde{c}_j d = 1$ holds. This also shows that if we compose $\tilde{\rho}$ by $\mathrm{Spin}_n \rightarrow \mathrm{SO}_n$ then we get the pull-back \tilde{E} of E . \square

The group $\tilde{\Gamma}$ defines a two-sheeted cover of $p_1 : \tilde{Y} \rightarrow Y$ which is ramified at the ramification points of $p : Y \rightarrow X$ and one additional point of Y and infact if $\mathbb{H}_{\tilde{Y}}$ denotes the simply connected cover of \tilde{Y} then $\tilde{\Gamma}$ is the deck transformation group of $\mathbb{H}_{\tilde{Y}} \rightarrow X$. The ambiguities in defining $\tilde{\rho}$ correspond to the different lifts to $\tilde{\pi}$ - Spin_n -bundles of \tilde{E} . We remark that $\tilde{\pi}$ contains the group $\mathbb{Z}/2 = \mathrm{Gal}(\tilde{Y}/Y)$ as a normal subgroup. This group acts on the lifts of \tilde{E} . But its action is not trivial along the fibers, so these bundles do not descend as Spin -bundles. However its image lies in the $\ker(\mathrm{Spin}_n \rightarrow \mathrm{SO}_n)$ and thus on \tilde{E} it acts trivially along the fiber. Thus \tilde{E} descends.

Proposition 3.1.2. Parabolic degenerate special orthogonal bundles can be obtained by extending structure group from *parahoric Spin-bundles* on X .

Proof. By Theorem 3.0.20, it suffices to prove that π - SO_n bundles on Y can be obtained from parahoric Spin bundles on Y by extending structure group $\mathrm{Spin}_n \rightarrow \mathrm{SO}_n$. We fix a maximal compact K of Spin_n . Let R_Y denote the set of branch points of $p_1 : \tilde{Y} \rightarrow Y$, which is also equal to the set of ramification points of $p : Y \rightarrow X$. For every $y \in R$,

we fix a conjugacy class C_y in $\mathrm{SO}_n(\mathbb{R})$ of finite order n_y . Note here that for two y_1, y_2 in the same fiber of p , we have the equality $C_{y_1} = C_{y_2}$. Let C_y^1 and C_y^2 be two conjugacy classes in K lifting C_y . The order of C_y^i may possibly be $2n_y$. For every function $f : R \rightarrow \{1, 2\}$, we denote

$$\mathrm{Rep}_f = \{\tilde{\Gamma} \rightarrow \mathrm{Spin}_n | \tilde{\rho}(\tilde{c}_y) \in C_y^{f(y)}, y \in R\} / \{\text{conjugation by } K\}$$

which by [2, Balaji-Seshadri, Thm 7.1] is irreducible. These correspond to $\tilde{\pi}$ - Spin_n bundles on \tilde{Y} such that by extending the structure group $\mathrm{Spin}_n \rightarrow \mathrm{SO}_n$, we get $\tilde{\pi}$ - SO_n bundles with trivial $\mathrm{Gal}(\tilde{Y}/Y)$ group action along the fibers. Thus these bundles descend to π - SO_n bundles on Y of fixed parabolic type. On the other hand, viewing these bundles as $\mathrm{Gal}(\tilde{Y}/Y)$ - Spin_n bundles on \tilde{Y} , they descend as *parahoric Spin-bundles on Y* ; moreover together they form all the lifts of π - SO_n bundles of fixed parabolic type to *parahoric Spin-bundles on Y* . Thus the parahoric spin bundles on X in the sense of [2] that come from Rep_f , as f varies, form all the lifts of parabolic degenerate special orthogonal bundles of fixed parabolic datum given by $\{C_y\}_{y \in R}$, which is actually a datum on X . \square

Remark 3.1.3. The Proposition 3.1.2 justifies that parabolic degenerate special orthogonal bundles may also be called as *parahoric special orthogonal bundles* and we shall do so in the rest of the paper.

4. PARAHORIC SYMPLECTIC BUNDLES

Definition 4.0.4. A degenerate symplectic bundle with flags is a vector bundle (V, q, F^\bullet, L) on X endowed with the datum

- (1) a symplectic form $q : V \rightarrow V^* \otimes L$ with singularities of order ≤ 1 at a finite subset R of points of X ,
- (2) a flag structure $\{0\} \subsetneq F_x^{m_x} \subsetneq F_x^{m_x-1} \subsetneq \dots \subsetneq F_x^1 \subsetneq F_x^0 = V_x$ for each point $x \in R$, where the number m_x can vary with $x \in R$,

satisfying the conditions

- (1) (compatibility of symplectic form q and the flags) for every $x \in R$ we have $F_x^1 = \mathrm{Ker}(q_x : V_x \rightarrow (V^* \otimes L)_x)$ and dimension of F_x^1 is even.
- (2) we denote by $F^1(V)$ the vector bundle obtained by pulling back $0 \rightarrow V(-R) \rightarrow V \rightarrow \bigoplus_{x \in R} V_x \rightarrow 0$ by $\bigoplus_{x \in R} F_x^1 \hookrightarrow \bigoplus_{x \in R} V_x$.

Then q restricted to $F^1(V)$ factorizes as q_1 through $L(-R)$:

$$\begin{array}{ccc} F^1(V) \times F^1(V) & \xrightarrow{q} & L \\ & \searrow q^1 & \uparrow \\ & & L(-R) \end{array}$$

- (3) the symplectic form q^1 induces a non-degenerate symplectic form

$$q_{1,x} : F_x^1 \rightarrow L(-R)_x \simeq \mathbb{C}.$$

- (4) for $i \geq (1+m_x)/2$, the flags F_x^i are Lagrangian for $(F_x^1, q_{1,x})$ and the remaining can be obtained by symplecto-complementation.

Remark 4.0.5. The compatibility of the global symplectic form and the local ones can be expressed as follows: if we choose a basis \mathfrak{B} of the localization of the free module V_x it induces a basis of V_x in which q_i 's can be expressed by square matrices of sizes $\dim(F_x^1)$ and $n - \dim(F_x^1)$ in the form

$$J' = \begin{bmatrix} & & & & 1 \\ & & & -1 & \\ & & \ddots & & \\ & 1 & & & \\ -1 & & & & \end{bmatrix}$$

then in terms of \mathfrak{B} the quadratic form q_x can be brought to the form $\begin{bmatrix} J'_1 \\ tJ'_2 \end{bmatrix}$ where t is the local parameter of L_x . This special form is chosen to be able to Hecke-modify by Lagrangian subspaces (cf [13]).

Remark 4.0.6. As a consequence of the definition, we have a non-degenerate symplectic form $q_{0,x} : G^0 = V_x/F^1 \rightarrow \mathbb{C}$. Notice that on successive quotients $G_x^i = F_x^i/F_x^{i+1}$ we have perfect pairings $q_{i,x} : G_x^{m-i} \times G_x^i \rightarrow \mathbb{C}$ for $1 \leq i \leq m_x/2$ and on quotients G^0 and $G^{(1+m_x)/2}$ (if m_x is odd) we have non-degenerate symplectic forms $q_{0,x}$ and $q_{(1+m_x)/2,x}$. The spaces G^0 and $G^{(1+m_x)/2}$ are constrained therefore to be *even* dimensional unlike the orthogonal group case (compare with Remarks 2.0.4 and 3.0.22). Conversely, G_x^i for $i \geq 1$ will endow F_x^1 with a non-degenerate symplectic form. Together with G_x^0 , they endow the stalk V_x with a degenerate symplectic form of the desired type.

Like Theorem 2.0.14, one similarly proves the following theorem.

Theorem 4.0.7. Let W be a π -GL $_n$ bundle on Y such that $W(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}) \rightarrow Y$ admits a π -invariant section q' . Then to such a bundle we can canonically associate a parabolic degenerate symplectic bundle $(V, q, F^\bullet, \mathcal{O}_X)$

with flags at precisely the branch points of $p : Y \rightarrow X$ and weights symmetric about $1/2$. Conversely, given $(V, q, F^\bullet, \mathcal{O}_X)$ a parabolic degenerate symplectic bundle with flags on a smooth projective curve X with weights symmetric about $1/2$, there exists a Galois cover $p : Y \rightarrow X$ with Galois group π along with a vector bundle W on Y and a π -invariant section $q' : Y \rightarrow W(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n})$ such that (W, q') is mapped to (V, q) by the first part of the theorem.

Proof. The proof is similar to that of Theorem 2.0.14. We need only check that $\dim F_x^1$ is even. One knows that π_y acts on the fiber of the symplectic bundle (W, q') through a representation $\rho_y : \pi_y \rightarrow \mathrm{Sp}_{2n}$ (cf [7, Prop 1, page 06]). Since π_y is cyclic, so its image lands inside a maximal torus of Sp_{2n} . This implies that dimension F_x^1 is even. \square

Remark 4.0.8. By Theorem 4.0.7, the degenerate symplectic bundles with flags correspond to π -Sp-bundles on some Galois cover which by [2] descend as parahoric symplectic bundles on X . We shall therefore call degenerate symplectic bundles with flags simply as parahoric symplectic bundles in the rest of the paper.

5. CRITERION OF NON-EMPTYNESS OF PARAHORIC MODULI ON \mathbb{P}^1 FOR $G = \mathrm{SO}_n, \mathrm{Sp}_{2n}$

5.1. (Semi)-stability conditions. In this subsection we shall admit that (semi)-stability of parahoric G bundles is an open property. This fact follows from the so called Γ -bundle theory since stable bundles correspond to irreducible representations of Γ into the maximal compact subgroup K of G (cf. [1]). Here Γ is the decktransformation group over X of the simply connected cover \tilde{Y} of Y .

Definition 5.1.1. Let (V, q, L) be a parahoric orthogonal bundle. We say that a sub-bundle W of V is isotropic if for all $x \in X \setminus R$, the fiber $W_x \subset V_x$ is an isotropic subspace and for $x \in R$, $W_x \cap F_x^1$ is an $q_{1,x}$ -isotropic subspace of F_x^1 and the image of W_x in V_x/F_x^1 is $q_{0,x}$ -isotropic subspace.

As in the case of Parabolic vector bundles, an isotropic sub-bundle W of V inherits a flags by intersecting $W_x \cap F_x^\bullet$ and also weights.

Definition 5.1.2. We say that a parahoric orthogonal or symplectic bundle (V, q, L) is (semi)-stable if for every isotropic sub-bundle W in the sense of Definition 5.1.1, the inequality of parabolic slope $\mathrm{par}\mu(W)(\leq)\mathrm{par}\mu(V)$ is satisfied.

Remark 5.1.3. Our definition 5.1.2 agrees with the one in [5] which was made for parahoric bundles ‘comming from’ two-sheeted covers.

To see that this definition of (semi)-stability is correct, firstly it suffices to assume that $L = \mathcal{O}_X$ because we could always take the square root of L (if need be by going to a cover), which doesn't change the (semi)-stability property of the bundles we consider. Next, the bundle (V, q, F^\bullet) by Theorem 2.0.14 and 4.0.7 would become genuine orthogonal or symplectic bundles (W, q') on some Galois cover $p : Y \rightarrow X$ with Galois group π . The pull back of isotropic sub-bundles with induced weights in the sense of Definition 5.1.1, would give π -sub-bundles of W isotropic for q' and conversely any q' -isotropic sub-bundle of W to which the π -linearization restricts well would give an isotropic sub-bundle of (V, q) in the sense of Definition 5.1.1. Thus the condition of (semi)-stability for parahoric orthogonal (symplectic) bundles is translated into π -isotropic bundles, for which the (semi)-stability condition is the usual condition for G -bundles (cf. [17, A.Ramanathan]) applied to every π -reduction of structure group to parabolic subgroups (cf. [1, Def. 6.4.6]).

5.2. Passage from Parahoric to Parabolic. In the following proposition, notice that in the case m_x is odd, the length of the Flag has increased by one, but in the even case, it remains the same. It is readily checked.

Proposition 5.2.1. Let $(V, q, \{F_x^\bullet, \alpha_x^\bullet\}_{x \in R})$ be a parahoric special orthogonal bundle on \mathbb{P}^1 of parabolic degree zero. For every $x \in R$, if m_x is even, then $F_x^{1+m_x/2}$ is a maximal $q_{1,x}$ -isotropic subspace of F_x^1 and we choose it, else when m_x is odd, we shall make a choice of a maximal $q_{1,x}$ -isotropic subspace K_x of F_x^1 containing the subspaces F_x^i for $i \geq 1+m_x/2$. Let \tilde{V} be a vector bundle defined using Rees theorem 2.0.11 by the inclusion $K_x \hookrightarrow V_x$ for every $x \in R$. Then

- (1) the quadratic form q on V extends uniquely to a non-degenerate quadratic form \tilde{q} on \tilde{V} .
- (2) by the inclusion $V_x/K_x \hookrightarrow \tilde{V}_x$, define \tilde{F}_x^i as the image of F_x^i/K_x for $i \leq 1+m_x/2$. Then the flags \tilde{F}_x^i are \tilde{q}_x -isotropic. We define V_x/K_x as \tilde{F}_x^0 .
- (3) the flags $\{F_x^i\}$ for $i \geq 1+m_x/2$ define a filtration of K_x . We can take their pull-back \tilde{F}_x^i to \tilde{V}_x by the projection $\tilde{V}_x \rightarrow K_x$. Then in the order of inclusion, the $\{\tilde{F}_x^i\}$ form an orthogonal grassmannian i.e the ortho-complement of the j -th smallest subspace is the j -th largest.
- (4) a sub-bundle W of V defines by Hecke-modification $W_x \cap K_x \hookrightarrow W_x$ a sub-bundle \tilde{W} of \tilde{V} . Then W is isotropic in the sense of

Definition 5.1.1 if and only if \tilde{W} is isotropic with respect to \tilde{q} in the usual sense.

- (5) the parabolic orthogonal bundle $(\tilde{V}, \tilde{q}, \tilde{F})$ is a parabolic special orthogonal bundle.
- (6) V is (semi)-stable as a parahoric orthogonal bundle if and only if the parabolic orthogonal bundle $(\tilde{V}, \tilde{q}, \tilde{F}^\bullet)$ supports a (semi)-stable parabolic orthogonal (resp. symplectic) structure with respect to the following definition of (semi)-stability

Definition 5.2.2. If $\tilde{F}_x^i \subset \ker(\tilde{V}_x \rightarrow K_x)$, then we assign it the weight $\tilde{\alpha}_x^i = \alpha_x^i$ where α_x^i is assigned to the inverse image of \tilde{F}_x^i under $V_x \rightarrow \tilde{V}_x$. Else, we assign it weight $\tilde{\alpha}_x^i = \alpha_x^i - 1$ where α_x^i is assigned to the image of \tilde{F}_x^i in $\tilde{V}_x \rightarrow K_x$.

We define the parabolic degree of a sub-bundle \tilde{W} of \tilde{V} as $\text{pardeg}(\tilde{W}) =$

$$\text{deg}(\tilde{W}) + \sum_{x \in R} \sum_{1 \leq i \leq m_x} \tilde{\alpha}_x^i \dim(\text{Img}(\tilde{W}_x \cap \tilde{F}_x^i \rightarrow \tilde{F}_x^i / \tilde{F}_x^{i+1})).$$

We say that the parabolic orthogonal bundle \tilde{V} is (semi)-stable if

$$\text{pardeg}(\tilde{W}) / \text{rank}(\tilde{W}) (\leq) \text{pardeg}(\tilde{V}) / \text{rank}(\tilde{V}).$$

- (7) For any isotropic sub-bundle W of V , the parabolic degree of W and \tilde{W} are the same.

For the convenience of the reader we make some remarks to clarify the effect of Hecke-modification by K_x on flags and weights.

Remark 5.2.3. It is also clear that \tilde{V} comes along with projection maps $\tilde{V} \rightarrow \bigoplus_{x \in R} K_x$ from which by taking kernels, it is possible to recover $(V, q, F^\bullet, \alpha^\bullet)$ from \tilde{V} .

Remark 5.2.4. We see that \tilde{V}_x has weight $-1/2$ when m_x is even and weight $\alpha_x^{(1+m_x)/2} - 1$ if m_x is odd.

Remark 5.2.5. We also see that if $\tilde{F}_x^1 \neq \tilde{F}_x^{1\perp} = \ker(\tilde{V}_x \xrightarrow{\tilde{q}_x} \tilde{V}_x^* \rightarrow \tilde{F}_x^{1*})$, then $\tilde{F}_x^{1\perp}$ is assigned weight zero because it is the image of $V_x / K_x \hookrightarrow \tilde{V}_x$ and V_x is assigned weight zero in this case.

Remark 5.2.6. We shall always consider the \tilde{F}_x^i in the order of inclusion and not by the index i , which has got disturbed. The index i is convenient to assign weights $\tilde{\alpha}_x^i$ using the weights α_x^i . Under this order, we see that in Definition 5.2.2 the parabolic weights are decreasing with the subspace becoming bigger in accordance with the definition in [11, Mehta-Seshadri].

Remark 5.2.7. Since the weights $\{\alpha_\bullet^i\}$ are symmetric about half, so the weights $\{\tilde{\alpha}_\bullet^i\}$ are *symmetrically distributed about zero*.

Remark 5.2.8. By choosing a K_x (if m_x is odd), one replaces the graded piece $(G_x^{(1+m_x)/2}, q_{(1+m_x)/2, x})$ by the perfect pairing $F_x^{(1+m_x)/2}/K_x \times K_x/F_x^{(1+m_x)/2+1} \rightarrow \mathbb{C}$.

Proof. After Remark 2.0.5 the first four assertions are just local checks at x . For see the first, let us suppose that the form q_x is represented as $x_1x_i + x_2x_{i-1} + \cdots + x_{i/2}x_{i/2+1}$ (or $x_{(1+i)/2}^2$) + $t(x_{i+1}x_n + x_{i+2}x_{n-1} + \cdots + x_{(n+i)/2}x_{(n+i)/2+1})$ in terms of the basis $\{e_i\}$ of the stalk at x of the locally free module V_x . Then after Hecke modification, if $\{e'_i\}$ denote the basis of \tilde{V}_x , then we have absorbed $te'_j = e_j$ for $j \geq (n+i)/2$ (one knows that $n-i$ is being the dimension of F_x^1 is even) to get $\tilde{q}_x = x_1x_i + x_2x_{i-1} + \cdots + x_{i/2}x_{i/2+1}$ (or $x_{(1+i)/2}^2$) + $(x_{i+1}x_n + x_{i+2}x_{n-1} + \cdots + x_{(n+i)/2}x_{(n+i)/2+1})$, which is non-degenerate. Notice that when m_x is even, we have the following

$$\{0\} \subsetneq \tilde{F}_x^{m_x/2} \subsetneq \cdots \subsetneq \tilde{F}_x^1 \subsetneq \tilde{F}_x^0 \subsetneq \tilde{F}_x^{m_x} \subsetneq \cdots \subsetneq \tilde{F}_x^{1+m_x/2} = \tilde{V}_x$$

and when m_x is odd, we have

$$\{0\} \subsetneq \tilde{F}_x^{(1+m_x)/2} \subsetneq \cdots \subsetneq \tilde{F}_x^1 \subsetneq \tilde{F}_x^0 \subsetneq \tilde{F}_x^{m_x} \subsetneq \cdots \subsetneq \tilde{F}_x^{(m_x-1)/2} \subsetneq \tilde{V}_x.$$

Here again we see that in the case m_x is odd, the length of the Flag has increased by one because $\tilde{F}_x^{(m_x-1)/2} \subsetneq \tilde{V}_x$, (but in case m_x is even we have $\tilde{F}_x^{[(1+m_x)/2]} = \tilde{V}_x$).

For the next assertion, recall that \tilde{V} fits into the short exact sequence

$$(5.2.1) \quad 0 \rightarrow V \rightarrow \tilde{V} \rightarrow \oplus_{x \in R} K_x \rightarrow 0$$

Let $O_q^c \rightarrow X$ denote the group scheme of the completed parahoric orthogonal bundle (V, q, F^\bullet) with K_x (if m_x is odd). The operation of modification that we have described corresponds (cf [1, Section 5.3 Hecke-correspondences]) to lifting (V, q, F) to the completed flags

$$\begin{array}{ccc} & Bun_X(O_q^c) & \\ \swarrow & & \searrow \\ Bun_X(O_q) & & Bun_X(O_{\tilde{q}}) \end{array}$$

which is always possible since $Bun_X(O_q^c) \rightarrow Bun_X(O_q)$ is a projective morphism and then taking the image by the other arrow. In other words, by 5.2.1 it follows that the local automorphisms of V

as a parahoric orthogonal bundle that further respect K_x on the special fiber and the associated perfect pairings (cf Remark 5.2.8), furnish local automorphism of \tilde{V} (the proof is similar to the proof of Prop 2.0.17). Since we work with a parahoric special orthogonal bundle $(V, q, F, s_X) \in Bun_X(\mathrm{SO}_q)$ and $Bun_X(\mathrm{SO}_q)$ is a component of $Bun_X(O_q)$ so the lift lies in the component $Bun_X(\mathrm{SO}_q^c)$ and therefore after Hecke-modification, $(\tilde{V}, \tilde{q}, \tilde{F}^\bullet)$ lies in $Bun_X(\mathrm{SO}_{\tilde{q}})$.

The next assertion is also only a check. Interpreting the parabolic degree as in Definition 5.1.2 of $W \subset V$ *intrinsically* in terms of \tilde{W} we get

$$\deg(\tilde{W}) + \sum_{x \in R} \sum_{1 \leq i \leq m_x} \alpha_x^i \dim(\mathrm{Img}(\tilde{W}_x \cap \tilde{F}_x^i \rightarrow \tilde{F}_x^i / \tilde{F}_x^{i+1})) - \dim(\mathrm{Img}(\tilde{W}_x \hookrightarrow \tilde{V}_x \rightarrow K_x)).$$

Now $-\dim(\mathrm{Img}(\tilde{W}_x \hookrightarrow \tilde{V}_x \rightarrow K_x)) = \dim(\tilde{W}_x \cap \mathrm{Img}(V_x/K_x \hookrightarrow \tilde{V}_x)) - \mathrm{rank}(\tilde{W})$. The term $\dim(\tilde{W}_x \cap \mathrm{Img}(V_x/K_x \hookrightarrow \tilde{V}_x))$ can be accounted for by defining parabolic degree as

$$\deg(\tilde{W}) + \sum_{x \in R} \sum_{1 \leq i \leq m_x} \alpha_x'^i \dim(\mathrm{Img}(\tilde{W}_x \cap \tilde{F}_x^i \rightarrow \tilde{F}_x^i / \tilde{F}_x^{i+1}))$$

and replacing the weights α_x^i by $\alpha_x'^i$ defined as

$$\begin{aligned} \alpha_x^i + 1 & \quad \text{if } \tilde{F}_x^i \subset \ker(\tilde{V}_x \rightarrow K_x) \\ \alpha_x^i & \quad \text{if otherwise} \end{aligned}$$

Now the weights $\alpha_x'^i$ belong to the interval $[1/2, 3/2]$. The term $-\mathrm{rank}(\tilde{W})$ can be accounted for by decreasing all the weights by one. The sliding of weights does not affect the (semi)-stability properties. Now the new weights are exactly $\tilde{\alpha}_x^i$ of Definition 5.2.2 as desired. Now we also see that the parabolic degree has remained unchanged, as we have only interpreted that of W in terms of \tilde{W} .

□

Remark 5.2.9. One takes parabolic weights in the interval $[0, 1]$ as in [11, Mehta-Seshadri] to specify the ratios of polarisations on $\mathrm{Quot} \times \mathrm{Flagvarieties}$ for the purpose of GIT constructions. But for computational purposes of (semi)-stability it seems better to formulate the conditions with positive and negative weights, for then we can simply demand for sub-bundles that parabolic degree $(\leq) 0$ when the ambient bundle has parabolic degree zero (instead of the usual slope condition). This is the choice we have made in this paper, as in [3]. For GIT constructions, one can slide the weights for the purposes of GIT constructions. Sliding only changes the conditions, but not the (semi)-stability property satisfied or not by a bundle.

5.3. Passage to generic bundles. Recall for an orthogonal bundle (\tilde{V}, \tilde{q}) on \mathbb{P}^1 the *Mumford invariant* $\mu(\tilde{V})$ is defined as $h^0(\tilde{V}(-1)) \bmod 2$ (cf [12, Mumford, page 184]) which is invariant under deformations of orthogonal bundles.

Definition 5.3.1. We say that two orthogonal bundles E_0 and E_1 can be deformed into each other if there is a connected complex space T , an orthogonal bundle on $\mathbb{P}^1 \times T$ and two points $x, y \in T$ such that $E|_{\mathbb{P}^1 \times \{x\}} \simeq E_0$ and $E|_{\mathbb{P}^1 \times \{y\}} \simeq E_1$.

We also recall

Theorem 5.3.2. [9, Hulek] Two orthogonal bundles of rank at least 3 can be deformed into each other if and only if they have the same Mumford invariant.

Another reference for the above is A.Ramanathan [16, iii) of 9.5.1 and 9.5.2] for type B_l and D_l . This means that every orthogonal bundle on \mathbb{P}^1 is deformable to either the trivial bundle or $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{n-2}$. For the symplectic case, A.Ramanathan [16, 9.7, iii)] has proved that the trivial bundle on \mathbb{P}^1 is rigid. This means that any symplectic bundle can be deformed to the trivial bundle.

In [6] Grothendieck proved that a vector bundle \tilde{V} on \mathbb{P}^1 has upto isomorphism at most one structure as an orthogonal bundle. Thus the obvious necessary condition $\tilde{V} \simeq \tilde{V}^*$ also becomes sufficient.

Proposition 5.3.3. The parabolic special orthogonal bundle $(\tilde{V}, \tilde{q}, \tilde{F}_x^\bullet, \alpha_x^\bullet)$ is (semi)-stable if and only if the bundle

$$\begin{aligned} & \mathcal{O}_{\mathbb{P}^1}^n \quad \text{if } \mu(\tilde{V}) = 0 \\ & \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2} \quad \text{if } \mu(\tilde{V}) = 1 \end{aligned}$$

endowed with generic parabolic structure of type $(\tilde{F}_x^\bullet, \tilde{\alpha}_x^\bullet)$ is (semi)-stable. A parabolic symplectic bundle $(\tilde{V}, \tilde{q}, \tilde{F}_x^\bullet, \alpha_x^\bullet)$ is (semi)-stable if and only if the trivial bundle with generic symplectic parabolic structure is (semi)-stable.

Proof. By Hulek's and Ramanathan's theorems, it follows that (\tilde{V}, \tilde{q}) can be put in a T -family over \mathbb{P}^1 where the generic member V_{gen} is the trivial bundle or $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{n-2}$ depending upon the Mumford invariant in the orthogonal case and the trivial bundle in the symplectic case. Since G -bundles are locally isotrivial, so for every parabolic point $w \in R$, there is a non-empty open subset $T_w \subset T$ and a neighbourhood U_w of w , such that the restriction of (\tilde{V}, \tilde{q}) to $U_w \times T_w$ is trivial. Without loss of generality, we may assume that T is irreducible and hence $T_w \subset$

T are dense open subsets. Thus on the intersection $\cap_{w \in R} T_w \subset T$ which is non-empty open and dense, the flags $\{F_w^\bullet\}$ can be extended for every $w \in R$. They can be endowed with the same weights. Thus replacing T by $\cap_{w \in R} T_w$, we see that $(\tilde{V}, \tilde{q}, \tilde{F}_x^\bullet, \tilde{\alpha}_x^\bullet)$ can be put in a family of *parabolic* orthogonal bundles endowed with parabolic structure of type $(\tilde{F}_x^\bullet, \tilde{\alpha}_x^\bullet)$ where the vector bundle underlying a generic object splits is V_{gen} . In the following, we replace T by the connected component of $\cap_{w \in R} T_w$ containing \tilde{V} .

We first argue for the symplectic case as the group is simply connected.

The openness of (semi)-stability in a family is assured by the fact that parahoric symplectic bundles correspond to $\pi\text{-Sp}_{2n}$ bundles on some cover and for such bundles this follows by general Γ -bundle theory (cf. [1]). So the two open sets corresponding to \mathcal{P} such that its underlying bundle is V_{gen} and to (semi)-stable \mathcal{P} must intersect since $\text{Bun}_{\mathcal{G}}$ is irreducible. It follows that the bundle V_{gen} for a generic Lagrangian flag supports a (semi)-stable parahoric symplectic structure.

For the case of parahoric special orthogonal bundles, we have to argue a little more because SO_n is not simply connected.

To complete the proof we introduce some notation from [8].

Let \mathcal{G}_X denote the ‘parahoric for SO_n at the parabolic points’ Bruhat–Tits group scheme associated to parahoric special orthogonal bundles and let $\tilde{\mathcal{G}}_X$ be its lift to the ‘parahoric for *Spin*’ type Bruhat–Tits group scheme. The way to do this is explained on [8, page 513]. Let $\mathcal{Z}^{fin} \rightarrow X$ denote the kernel group scheme of the morphism $\tilde{\mathcal{G}} \rightarrow \mathcal{G}$. Now $\text{Bun}_{\mathcal{G}}$ is again disconnected and its components are parametrized by $H^2(X, \mathcal{Z}^{fin})$ by [8, Lemma 14, part (4) applied to (3) and Lemma 15], which for our purposes is a certain quotient of $H^2(X, \mathcal{Z}^{fin})$ and hence *finite*. Each of its connected components is isomorphic to the quotient of $\text{Bun}_{\tilde{\mathcal{G}}}$ under the action of $H^1(X, \mathcal{Z}^{fin})$. Again since $\text{Bun}_{\tilde{\mathcal{G}}}$ is smooth, so this quotient is irreducible. This quotient must contain a \mathcal{G} -torsor whose underlying bundle is V_{gen} or else it will be a union of orbits of non-trivial bundles whose orbits we know are of strictly lesser dimension than that of $\text{Bun}_{\mathcal{G}}$. Thus \mathcal{G} -torsors whose underlying bundle is actually V_{gen} will form an open dense subset. Now we can conclude as in the symplectic case. \square

5.4. Recall of Schubert states and Gromov–Witten numbers.

We recall that R denotes the set of parabolic points. For $w \in R$, we consider generic complete orthogonal grassmanian G_w^\bullet on \tilde{V}_w .

For a subset $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$, define the Schubert variety

$$\Omega_I^O(G^\bullet) = \{L \in Gr(r, \tilde{V}_w) \mid \dim(L \cap G^{i_j}) \geq j \text{ for all } 1 \leq j \leq r\}.$$

Definition 5.4.1. Let $Gr(r, n)$ denote the Grassmanian of r -dimensional isotropic subspaces of a n -dimensional vector space with a non-degenerate quadratic form. For subsets $I_w \subset \{1, \dots, n\}$ of cardinality r we denote by $\langle \sigma_{I_w} \rangle_{w \in R} >_d$ Gromov–Witten numbers defined as the number of maps $f : \mathbb{P}^1 \rightarrow Gr(r, n)$ of degree d such that for $w \in R$ we have $f(w) \in \Omega_{I_w}^O(G_w^\bullet)$.

The degree d maps from $\mathbb{P}^1 \rightarrow Gr(r, n)$ correspond bijectively to isotropic sub-bundles of rank r and degree $-d$ of the trivial bundle (endowed with the unique orthogonal structure upto isomorphism by Grothendieck’s theorem [6]). The Gromov–Witten number counts therefore the number of isotropic sub-bundles W of the trivial bundle of degree $-d$ and rank r such that the fiber W_w , for $w \in R$ a parabolic point, lies in the Schubert variety $\Omega_{I_w}^O(G_w^\bullet)$.

We now describe a slight generalisation of Gromov–Witten numbers (for more details cf. also [4, Sections 1.5 and 3]) to also treat the bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{n-2}$. So more generally let W be a vector bundle on \mathbb{P}^1 such that $W^* \simeq W$. Define $Gr(d, r, W)$ to be the moduli space of isotropic sub-bundle of W of rank r and degree d . For $p \in \mathbb{P}^1$, define projection maps $\pi_p : Gr(d, r, W) \rightarrow Gr(r, W_p)$ to the fiber of W at p . We call a Schubert State $\mathfrak{J} = (d, r, W, \{I_w\}_{w \in R})$ where $I_w \subset \{1, \dots, n\}$ of cardinality r and d is an integer. For a Schubert state \mathfrak{J} define $\langle \mathfrak{J} \rangle$ to be the number of points in the intersection (if finite and 0 otherwise)

$$\Omega^O(\mathfrak{J}, W, G^\bullet) = \cap_{w \in R} \pi_w^{-1}[\Omega_{I_w}^O(G_w^\bullet)] \subset Gr(d, r, W).$$

In the case W is the trivial bundle, the number $\langle \mathfrak{J} \rangle$ corresponds to the usual Gromov–Witten invariants. It counts the number of isotropic sub-bundles U of degree $-d$ and rank r of W such that the fiber U_w lies in the Schubert variety $\Omega_{I_w}^O(G_w^\bullet)$.

Remark 5.4.2. In [3], if $\dim(\mathfrak{J}) \neq 0$ then one defines $\langle \mathfrak{J} \rangle = 0$. We shall not do so to be able to handle stability.

Remark 5.4.3. We wish to explain the relevance of Gromov–Witten number being one in the context of semi-stability. If it is more than one or infinity for some Schubert state, then the associated isotropic sub-bundles can never be destabilizing in the sense of Harder–Narasimhan for any choice of weights, as they will have the same rank and the same parabolic degree which are completely determined by the Schubert states. In fact for any choice of weights, there will always be a

even more destabilizing isotropic sub-bundle by the existence assertion in Harder-Narasimhan theorem. . Hence to check semi-stability, we can restrict our attention to those of GW number one. On the other hand, if for some choice of weights, a particular sub-bundle is maximal destabilizing then its GW number must be one by definition of maximal destabilizing.

5.5. Formulation of inequalities. We refer the reader to Proposition 5.2.1 and 5.3.3 for the notations. In particular $\tilde{\alpha}_x^\bullet$ are deduced from α_x^\bullet as in Definition 5.2.2.

Let $\lambda_{I_w}(\tilde{\alpha}_w^\bullet)$ denote $\sum_{i \in I_w} \tilde{\alpha}_w^i$.

Theorem 5.5.1. There exists a semi-stable (resp. stable) parahoric special orthogonal bundle with parabolic datum $\{F_w^\bullet, \alpha_w^\bullet\}_{w \in R}$ if and only if either of the following conditions holds

- (1) given any $1 \leq r \leq n/2$ and any choice of subsets $\{I_w\}_{w \in R}$ of cardinality r of $\{1, \dots, n\}$, whenever $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_d = 1$ then $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d \leq 0$.
- (2) Let $W = \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{n-2}$. For every Schubert State $\mathfrak{J} = (d, r, W, \{I_w\}_{w \in R})$, whenever $\langle \mathfrak{J} \rangle = 1$, then for $I_w \in \mathfrak{J}$, we should have $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d \leq 0$.

Similarly, for stability either of the following conditions should hold

- (1) whenever $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_d \neq 0$ or is ∞ then $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d < 0$.
- (2) whenever $\langle \mathfrak{J} \rangle \neq 0$ or is ∞ , then for $I_w \in \mathfrak{J}$, we should have $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d < 0$.

Proof. Without loss of generality we may assume that the parabolic degree of the ambient bundle is zero. We first deal with semi-stability. By Proposition 5.2.1 the bundle $(V, q, F^\bullet, \alpha^\bullet)$ is semi-stable if and only if $(\tilde{V}, \tilde{q}, \tilde{F}^\bullet, \tilde{\alpha}^\bullet)$ is semi-stable, which by Proposition 5.3.3 and its proof is semi-stable in the sense of Definition 5.2.2 if and only if the vector bundle V_{gen} is semi-stable as a parahoric special orthogonal bundle with parabolic weights $\tilde{\alpha}_w^\bullet$ on generic orthogonal flags at points $p_i \in R$.

Suppose that the inequality corresponding to $\{\sigma_{I_w}\}_{w \in R}$ for $|I_j| = r$ and degree d is violated. We can move to generic flags at p_i by the openness of semi-stability. Since the intersection number $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_d \neq 0$, so we will find an isotropic sub-bundle W of rank r and degree d of a (semi)-stable parabolic bundle V_{gen*} , with underlying bundle V_{gen} , such that the fibers of W at p_i are in $\Omega_{I^\bullet}^{\mathcal{O}}(G^\bullet)$. Moreover it is readily checked that the parabolic degree of W is exactly $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d$. The violation of the inequality means that V_{gen*} is not semi-stable, which means that the original bundle was itself not semi-stable.

Conversely suppose that the inequalities are valid, but that V_{gen} with parabolic special orthogonal structure on generic flags, denoted V_{gen*} , is not (semi)-stable in the sense of Definition 5.2.2. We fix a generic parahoric structure. Since we have proved in Theorem 2.0.14 that parahoric orthogonal bundles correspond to π - O_n , so Harder-Narasimhan filtrations exist for parahoric orthogonal bundles. Thus HN filtrations also exist for parahoric special orthogonal bundles. Let W be the unique destabilizing isotropic sub-bundle of V_{gen*} of degree d_W and rank r . Let I_w be the set of cardinality r consisting of i_k where i_k is the least number in $\{1, \dots, n\}$ such that $\dim(G_w^\bullet \cap W_w)$ is k , as k varies from 1 to r . Note that the Gromov–Witten number $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_{d_W} = 1$ because firstly W exists and secondly because there is no sub-bundle M of degree d_W and rank r with fibers in $\Omega_{I_w}^O(G_w^\bullet)$ for $w \in R$. If such a M were to exist then we would have $pardeg(M) \geq pardeg(W)$. Now the uniqueness conclusion in the Harder-Narasimhan theorem implies that $pardeg(M) = pardeg(W)$, this forces $M = W$ since both have the same rank r . Since the intersection number is one, so in the set of inequalities there is one that corresponds to $\{I_w\}_{w \in R}$. Now $pardeg(W) > 0$ contradicts that inequality.

For stability notice that as one varies over the choices of Schubert states on parabolic points, the parabolic degree and therefore the parabolic slope remain invariant. Now the assertion follows owing to the fact that the Gromov–Witten numbers are computable if they are finite and algorithmically one can know if they are infinite. \square

Similarly we get

Theorem 5.5.2. There exists a semi-stable parahoric symplectic bundle with parabolic datum $\{F^\bullet, \alpha^\bullet\}_{w \in R}$ if and only if given any $1 \leq r \leq n/2$ and any choice of subsets $\{I_w\}_{w \in R}$ of cardinality r of $\{1, \dots, n\}$, whenever $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_d = 1$ then $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d(\leq) 0$. Similarly, for stability whenever $\langle \{\sigma_{I_w}\}_{w \in R} \rangle_d \neq 0$ or is ∞ then $\sum_{w \in R} \lambda_{I_w}(\tilde{\alpha}_w^\bullet) - d < 0$.

Remark 5.5.3. The Gromov–Witten numbers are computable for any G/P where P is any parabolic subgroup (cf. introduction to [19, Teleman-Woodward] where the reference given is [20, Woodward]).

The following proposition is a slight generalisation of a proposition of Ramanathan [15, Prop 7.1]. For the sake of completeness we give the proof because though π -semi-stability is equivalent to semi-stability but π -stability is *weaker* than stability.

Proposition 5.5.4. Let G and H be reductive algebraic groups and $\phi : G \rightarrow H$ be a surjective homomorphism. Let E be a π - G bundle

and E' the π - H -bundle obtained by extension of structure group by ϕ . Then if E' is π -stable (resp π -semi-stable) then E is π -stable (resp π -semi-stable). If further $N = \ker \phi \subset Z$ then conversely if E is π -stable (resp. π -semi-stable) then E' is π -stable (resp. π -semi-stable).

Proof. Suppose that E' is π -stable and that E is not. Then E admits a π -reduction to a maximal parabolic subgroup P such that the stability condition is violated. This reduction gives a π -reduction of E' to $\phi(P)$, which would be a maximal parabolic subgroup of H , violating the stability conditions for E' . Conversely, suppose that E is π -stable and E' has a π -reduction to a maximal parabolic P' of H . We put $P = \phi^{-1}(P')$, which is a maximal parabolic subgroup of G . We wish to show that there is a π -reduction of E to P giving the π -reduction of E' to P' . By the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & P' & \longrightarrow & 1 \end{array}$$

where the leftmost vertical arrow is an isomorphism, we get

$$\begin{array}{ccccccc} H^1(Y, \pi, N) & \longrightarrow & H^1(Y, \pi, G) & \longrightarrow & H^1(Y, \pi, H) & \longrightarrow & H^2(Y, \pi, N) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H^1(Y, \pi, N) & \longrightarrow & H^1(Y, \pi, P) & \longrightarrow & H^1(Y, \pi, P') & \longrightarrow & H^2(Y, \pi, N) \end{array}$$

where the first and the last vertical arrows are isomorphisms. Since E' arises from $E \in H^1(Y, \pi, G)$, so the π - P' bundle corresponding to the π -reduction of E' to P' comes from a $F \in H^1(Y, \pi, P)$. The group $H^1(Y, \pi, N)$ acts on $H^1(Y, \pi, P)$ and we can make the image of F to be E by acting by a suitable element. \square

Remark 5.5.5. By Proposition 5.5.4, the question of determining the existence of a (semi)-stable parahoric $Spin_n$ bundle reduces to the question of existence of a parahoric SO_n bundle, which has been answered by Theorem 5.5.1. For this we only have to note that the conjugacy classes of $Spin_n$ determine conjugacy classes of SO_n .

5.6. Cross-checks. For the convenience of the reader we recall

Theorem 5.6.1. [3, Thm 7, Belkale] Let $\{\overline{A_w}\}_{w \in R}$ be conjugacy classes in SU_n . Then there exists $A_w \in SU_n$ with conjugacy class $\overline{A_w}$ and $\prod A_w = \text{Id}$ if and only if given any $1 \leq r < n$ and any choice of subsets I_w of cardinality r and if $\langle \sigma_{I_w} \rangle_d = 1$ then $\sum_{w \in R} \lambda_{I_w}(\overline{A_w}) - d(\leq) 0$ holds.

The above theorem equivalently gives necessary and sufficient conditions for the existence of (semi)-stable parahoric SL_n -bundles. It is in this form that we shall use it in the following remarks that cross-check these conditions with those of Theorem 5.5.1 for the case of exceptional low rank isomorphisms.

Example 5.6.2. We wish to show that Belkale's conditions [3, Thm 7] agree with those in Theorem 5.5.1 for the case of exceptional homomorphism $\mathrm{SL}_2 \rightarrow \mathrm{SO}_3$. By Proposition 5.5.4, the question of existence of semi-stable parahoric SL_2 bundle is equivalent to that of existence of semi-stable parahoric SO_3 -bundle. Let V be a parahoric SL_2 -bundle on \mathbb{P}^1 . Let W be a parahoric sub-bundle of V (we could think of W as a genuine sub-bundle of V on a suitable cover). For simplicity let us first take the case of one parabolic point p and assume that we have weights $0 < \alpha_p < \beta_p < 1$ and one vector space $F_p \subset V_p$ as flag. Since the bundle is parahoric SL_2 , so $\alpha_p + \beta_p = 1$. We get the isotropic bundle, namely the image of $W \otimes_p W \hookrightarrow V \otimes_p V \rightarrow \mathrm{Sym}^2 V$ (here we take parabolic tensor product \otimes_p) denoted $\otimes_p^2 W$. By general theory one knows that $2\mathrm{pardeg} W = \mathrm{pardeg} \otimes_p^2 W$ and given an isotropic SO_3 -parahoric sub-bundle of $\mathrm{Sym}^2 V$ we can obtain a finite number of SL_2 -parahoric sub-bundle of V . Hence W is a maximal destabilizing sub-bundle of V if and only if $\otimes_p^2 W$ is a maximal destabilizing isotropic sub-bundle of $\mathrm{Sym}^2 V$. This takes care of the Gromov–Witten numbers being one in both cases. Recall that parahoric G -bundles correspond to representations of $\Gamma \rightarrow G$. So it follows that the weights of $\mathrm{Sym}^2 V$ would become $\alpha' = 2\alpha$ and $\beta' = 2\beta - 1$. In view of Remark 5.2.7 it suffices to work with them. Notice that we have $\alpha' + \beta' = 1$. Let us calculate the underlying degree of $\otimes_p^2 W$. Let $-d$ be the underlying degree of W . The parabolic degree of W is

$$\begin{aligned} -d + \beta_p & \text{ if } W_p \cap F_p \neq \{0\} \\ -d + \alpha_p & \text{ if } W_p \cap F_p = \{0\} \end{aligned}$$

The parabolic degree of $\otimes_p^2 W$ in the first case would be $2(-d + \beta_p) = -(2d-1) + \beta'_p$. This means that the underlying degree of $\otimes_p^2 W$ is $-(2d-1)$. Also $\mathrm{Sym}^2 V$ would endow $\otimes_p^2 W$ with a flag consisting of a single vector space of dimension one and weight β'_p . Similarly in the second case, the underlying degree is $-2d$ and the weight is α'_p . We recall that by Proposition 5.2.1 (7), we have $\mathrm{pardeg}(\otimes_p^2 W) = \mathrm{pardeg}(\otimes_p^2 \tilde{W})$, the latter being calculated by our conditions. Now conditions in Theorem 5.5.1 are just the conditions of [3, Thm 7] multiplied by the factor of two. For more than one parabolic point, we need only further remark

that the underlying degree of $\otimes_p^2 W$ is $-(2d - k)$ where k is the number of points $p \in R$ such that $W_p \cap F_p \neq \{0\}$. We see that after taking the parabolic tensor $\otimes_p^2 W$, the loss in weights is compensated by the gain in the underlying degree.

Example 5.6.3. We now wish to treat the case $\mathrm{SL}_4 \rightarrow \mathrm{SO}_6$. By Proposition 5.5.4, the question of existence of semi-stable parahoric SL_4 bundle is equivalent to that of existence of a semi-stable parahoric SO_6 -bundle. Let V be a parahoric SL_4 -bundle with a parahoric sub-bundle W . If W is a destabilizing line sub-bundle, then V admits a destabilizing quotient bundle of rank three. Thus $\Lambda_p^2 V$ admits a destabilizing quotient (hence sub-bundle) of rank three. On the other hand, if $\mathrm{rank} W = 2$ and destabilizing then we get $\Lambda_p^2 W$ a destabilizing line sub-bundle of $\Lambda_p^2 V$. If W is of rank 3 and destabilizing then we get a sub-bundle of $\Lambda_p^2 W \hookrightarrow \Lambda_p^2 V$ of rank 3. It follows from general theory there is a finite to one map from sub-bundles of V to parahoric sub-bundles of rank one and three of the second parabolic exterior bundle $\Lambda_p^2 V$. For the case of isotropic sub-bundles W_1 of $\Lambda_p^2 V$ of rank two, it can be seen by going to a suitable ramified cover (which is completely determined by the parabolic datum), that W_1 can be put in exactly two isotropic sub-bundles of rank three corresponding to the inverse image in W_1^\perp of the two isotropic line sub-bundles of W_1^\perp/W_1 , which is a rank two quadratic bundle. Hence isotropic sub-bundles of rank two do not correspond to reduction of structure group to maximal parabolics. Hence as far as (semi)-stability is concerned, we need to only consider sub-bundles of $\Lambda_p^2 V$ of rank one and three. If $\mathrm{rank} W = 2$ then, $\mathrm{pardeg} W = \mathrm{pardeg} \Lambda_p^2 W$, if $\mathrm{rank} W = 3$ then $2\mathrm{pardeg} W = \mathrm{pardeg} \Lambda_p^2 W$. Thus W is maximal destabilizing for V if and only if $\Lambda_p^2 W$ is maximal destabilizing for $\Lambda_p^2 V$. This takes care of the Gromov–Witten number being one. Suppose that V has weights $0 \leq \alpha^1 \leq \alpha^2 \leq \alpha^3 \leq \alpha^4 \leq 1$ (we allow weights to repeat as many times as their multiplicity for notational simplicity). If $\{\alpha^i\}$ are the weights appearing for W , then weights for $\Lambda_p^2 W$ are

$$\begin{aligned} \alpha^i + \alpha^j & \text{ if } 0 \leq \alpha^i + \alpha^j < 1 \\ \alpha^i + \alpha^j - 1 & \text{ if } 1 \leq \alpha^i + \alpha^j < 2 \\ \alpha^i + \alpha^j - 2 & \text{ if } 2 \leq \alpha^i + \alpha^j < 3. \end{aligned}$$

In view of Remark 5.2.7 it suffices to work with them. Similarly the underlying degree of $\Lambda_p^2 W$ is equal to the underlying degree of $\Lambda^2 W + j + 2k$ where j is the number of occurrences of case two and k of case

three. Recall that by Proposition 5.2.1 (7), we have $\text{pardeg}(\Lambda_p^2 W) = \text{pardeg}(\Lambda_p^2 \tilde{W})$, the latter being calculated by our conditions. In the case of $\text{rank} W = 2$ we see that conditions of [3, Thm 7] work out to the same for Theorem 5.5.1 and in case $\text{rank} W = 3$, they are exactly half.

Example 5.6.4. We treat the case $\text{Sp}_4 \rightarrow \text{SO}_5$. By Proposition 5.5.4, the question of existence of a semi-stable parahoric Sp_4 bundle is equivalent to that of existence of a semi-stable parahoric SO_5 -bundle. So let (V, q) be a rank 4 parahoric symplectic bundle. By going to a Galois cover $Y \rightarrow \mathbb{P}^1$, we see that $\Lambda_p^2 V$ will split as the direct sum $W \oplus \mathcal{O}_{\mathbb{P}^1}$, where W is a rank 5 parahoric orthogonal bundle. This is because on Y we would have $\mathcal{O}_Y \xrightarrow{q} \Lambda^2 V^* \xrightarrow{\Lambda^2 q^{-1}} \Lambda^2 V$. A destabilizing isotropic line sub-bundles L of V , would give a destabilizing rank 3 quotient Q , which upon taking the parabolic second exterior Λ_p^2 would give $\Lambda_p^2 V \rightarrow \Lambda_p^2 Q$. Now $\Lambda_p^2 Q$ is of rank 3 again, and so would give a rank two isotropic sub-bundle of W which would be destabilizing again. On the other hand, rank two isotropic sub-bundles U of V correspond to isotropic line sub-bundles $\Lambda_p^2 U$ of W . Now $\text{pardeg} U = \text{pardeg} \Lambda_p^2 U$, so U would be maximal destabilizing for V if and only if $\Lambda_p^2 U$ would be maximal destabilizing for W . This takes care of the Gromov–Witten numbers being one. For simplicity, we treat the case of one parabolic point p . If the weights at p of V are $\{\alpha^i\}$, by the description $(t_1, t_2, t_2^{-1}, t_1^{-1}) \mapsto (t_1 t_2, t_1 t_2^{-1}, 1, t_1^{-1} t_2, t_1^{-1} t_2^{-1})$ of the map between the maximal tori it follows that the weights of W would be

$$\begin{aligned} \alpha^i + \alpha^j & \text{ if } 0 < \alpha^i + \alpha^j < 1 \\ \alpha^i + \alpha^j - 1 & \text{ if } 1 \leq \alpha^i + \alpha^j < 2. \end{aligned}$$

In view of Remark 5.2.7 it suffices to work with them. The underlying degree of $\Lambda_p^2 U$ would be $\deg U$ in the first case and $\deg U + 1$ in the second. Recall that by Proposition 5.2.1 (7), we have $\text{pardeg}(\Lambda_p^2 U) = \text{pardeg}(\Lambda_p^2 \tilde{U})$, the latter being calculated by our conditions. We see that the conditions for (semi)-stability in Theorem 5.5.1 work out to the same for V and for W . The loss in weight is compensated by the gain in underlying degree. The case of many more than one parabolic points is similar.

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